Lecture 18
Iterative solution of linear systems*

Newton refinement
Conjugate gradient method
Review of Part II

Methods and Formulas

Basic Matrix Theory:

Identity matrix: \( AI = A, IA = A, \) and \( I \mathbf{v} = \mathbf{v} \)
Inverse matrix: \( AA^{-1} = I \) and \( A^{-1}A = I \)
Norm of a matrix: \( |A| \equiv \max_{\mathbf{v}} |A\mathbf{v}| \)
A matrix may be singular or nonsingular. See Lecture 10.

Solving Process:

Gaussian Elimination produces LU decomposition
Row Pivoting
Back Substitution

Condition number:

\[
\text{cond}(A) \equiv \max \left( \frac{\|\delta x\|/\|x\|}{\|A\|/|A| + \|\delta x\|/\|x\|} \right) = \max \left( \frac{\text{Relative error of output}}{\text{Relative error of inputs}} \right).
\]

A big condition number is bad; in engineering it usually results from poor design.

LU factorization:

\( PA = LU. \)

Solving steps:
Multiply by \( P \): \( \mathbf{d} = Pb \)
Forwardsolve: \( L\mathbf{y} = \mathbf{d} \)
Backsolve: \( U\mathbf{x} = \mathbf{y} \)

Eigenvalues and eigenvectors:

A nonzero vector \( \mathbf{v} \) is an eigenvector (ev) and a number \( \lambda \) is its eigenvalue (ew) if

\[ A\mathbf{v} = \lambda \mathbf{v}. \]

Characteristic equation: \( \det(A - \lambda I) = 0 \)
Equation of the eigenvector: \( (A - \lambda I)\mathbf{v} = \mathbf{0} \)

Complex ew’s:

Occur in conjugate pairs: \( \lambda_{1,2} = \alpha \pm i\beta \)
and \( \text{ev’s} \) must also come in conjugate pairs: \( \mathbf{w} = \mathbf{u} \pm i\mathbf{v} \).
Vibrational modes:
Eigenvalues are frequencies squared. Eigenvectors are modes.

Power Method:
- Repeatedly multiply $x$ by $A$ and divide by the element with the largest absolute value.
- The element of largest absolute value converges to largest absolute ev.
- The vector converges to the corresponding ev.
- Convergence assured for a real symmetric matrix, but not for an arbitrary matrix, which may not have real eigenvalues at all.

Inverse Power Method:
- Apply power method to $A^{-1}$.
- Use solving rather than the inverse.
- If $\lambda$ is an ev of $A$ then $1/\lambda$ is an ev for $A^{-1}$.
- The ev’s for $A$ and $A^{-1}$ are the same.

Symmetric and Positive definite:
- Symmetric: $A = A'$.
- If $A$ is symmetric its ev’s are real.
- Positive definite: $Ax \cdot x > 0$.
- If $A$ is positive definite, then its ev’s are positive.

QR method:
- Transform $A$ into $H$ the Hessian form of $A$.
- Decompose $H$ in $QR$.
- Multiply $Q$ and $R$ together in reverse order to form a new $H$.
- Repeat
- The diagonal of $H$ will converge to the ev’s of $A$.

Matlab
Matrix arithmetic:

```matlab
> A = [ 1 3 -2 5 ; -1 -1 5 4 ; 0 1 -9 0] ................. Manually enter a matrix.
> u = [ 1 2 3 4]'
> A*u
> B = [3 2 1; 7 6 5; 4 3 2]
> B*A ........................................................ multiply B times A.
> 2*A ........................................................ multiply a matrix by a scalar.
> A + A ........................................................ add matrices.
> A + 3 ........................................................ add 3 to every entry of a matrix.
> B.*B ........................................................ component-wise multiplication.
> B.^3 ........................................................ component-wise exponentiation.
```
Special matrices:

\[ \begin{align*}
&> I = \text{eye}(3) \quad \text{identity matrix} \\
&> D = \text{ones}(5,5) \\
&> O = \text{zeros}(10,10) \\
&> C = \text{rand}(5,5) \quad \text{random matrix with uniform distribution in } [0, 1]. \\
&> C = \text{randn}(5,5) \quad \text{random matrix with normal distribution.} \\
&> \text{hilb}(6) \\
&> \text{pascal}(5)
\end{align*} \]

General matrix commands:

\[ \begin{align*}
&> \text{size}(C) \quad \text{gives the dimensions } (m \times n) \text{ of } A. \\
&> \text{norm}(C) \quad \text{gives the norm of the matrix.} \\
&> \text{det}(C) \quad \text{the determinant of the matrix.} \\
&> \text{max}(C) \quad \text{the maximum of each row.} \\
&> \text{min}(C) \quad \text{the minimum in each row.} \\
&> \text{sum}(C) \quad \text{sums each row.} \\
&> \text{mean}(C) \quad \text{the average of each row.} \\
&> \text{diag}(C) \quad \text{just the diagonal elements.} \\
&> \text{inv}(C) \quad \text{inverse of the matrix.} \\
&> \text{C}' \quad \text{transpose of the matrix.}
\end{align*} \]

Matrix decompositions:

\[ \begin{align*}
&> [L \ U \ P] = \text{lu}(C) \\
&> [Q \ R] = \text{qr}(C) \\
&> [U \ S \ V] = \text{svd}(C) \quad \text{singular value decomposition (important, but we did not use it).} \\
&> H = \text{hess}(C) \quad \text{transform into a Hessian tri-diagonal matrix, which has the same eigenvalues as } A. \\
&> [U \ T] = \text{schur}(C) \quad \text{Schur Decomposition } A = U'TU. \\
&> R = \text{chol}(C'C) \quad \text{Cholesky decomposition of a symmetric, positive definite matrix, } A = R'R.
\end{align*} \]
Part III
Functions and Data