Lecture 18

Iterative solution of linear systems*

Newton refinement

Conjugate gradient method

Review of Part II

Methods and Formulas

Basic Matrix Theory:

Identity matrix: $AI = A$, $IA = A$, and $I\mathbf{v} = \mathbf{v}$
Inverse matrix: $AA^{-1} = I$ and $A^{-1}A = I$
Norm of a matrix: $\|A\| \equiv \max_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|$
A matrix may be singular or nonsingular. See Lecture 10.

Solving Process:

Gaussian Elimination produces LU decomposition
Row Pivoting
Back Substitution

Condition number:

$$\mathrm{cond}(A) \equiv \max \left(\frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}} \right) = \max \left(\frac{\text{Relative error of output}}{\text{Relative error of inputs}} \right).$$

A big condition number is bad; in engineering it usually results from poor design.

LU factorization:

$$PA = LU.$$

Solving steps:
Multiply by P: $\mathbf{d} = P\mathbf{b}$
Forwardsolve: $L\mathbf{y} = \mathbf{d}$
Backsolve: $U\mathbf{x} = \mathbf{y}$

Eigenvalues and eigenvectors:

A nonzero vector $\mathbf{v}$ is an eigenvector and a number λ is its eigenvalue if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Characteristic equation: $\det(A - \lambda I) = 0$
Equation of the eigenvector: $(A - \lambda I)\mathbf{v} = \mathbf{0}$
Residual for an approximate eigenvector-eigenvalue pair: $r = \|A\mathbf{v} - \lambda\mathbf{v}\|$

Complex eigenvalues:

Occur in conjugate pairs: $\lambda_{1,2} = \alpha \pm i\beta$
and eigenvectors must also come in conjugate pairs: $\mathbf{w} = \mathbf{u} \pm i\mathbf{v}$.

Vibrational modes:

Eigenvalues are frequencies squared. Eigenvectors represent modes.

Power Method:

- Repeatedly multiply $\mathbf{x}$ by A and divide by the element with the largest absolute value.
- The element of largest absolute value converges to largest absolute eigenvalue.
- The vector converges to the corresponding eigenvector.
- Convergence assured for a real symmetric matrix, but not for an arbitrary matrix, which may not have real eigenvalues at all.

Inverse Power Method:

- Apply power method to A^{-1}.
- Use solving rather than the inverse.
- If λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue for A^{-1}.
- The eigenvectors for A and A^{-1} are the same.

Symmetric and Positive definite:

- Symmetric: $A = A'$.
- If A is symmetric its eigenvalues are real.
- Positive definite: $A\mathbf{x} \cdot \mathbf{x} > 0$.
- If A is positive definite, then its eigenvalues are positive.

QR method:

- Transform A into H the Hessian form of A.
- Decompose H in QR.
- Multiply Q and R together in reverse order to form a new H.
- Repeat
- The diagonal of H will converge to the eigenvalues of A.

Matlab

Matrix arithmetic:

`A = [ 1  3 -2 5 ;  -1  -1 5 4 ; 0 1 -9  0]` .... Manually enter a matrix.
`u = [ 1  2  3  4]'`
`A*u`
`B = [3 2 1; 7 6 5; 4 3 2]`
`B*A` .... multiply B times A.
`2*A` .... multiply a matrix by a scalar.
`A + A` .... add matrices.
`A + 3` .... add 3 to every entry of a matrix.
`B.*B` .... component-wise multiplication.
`B.^3` .... component-wise exponentiation.

Special matrices:

`I = eye(3)` .... identity matrix
`D = ones(5,5)`
`O = zeros(10,10)`
`C = rand(5,5)` .... random matrix with uniform distribution in $[0,1]$.
`C = randn(5,5)` .... random matrix with normal distribution.
`hilb(6)`
`pascal(5)`

General matrix commands:

`size(C)` .... gives the dimensions $(m \times n)$ of A.
`norm(C)` .... gives the norm of the matrix.
`det(C)` .... the determinant of the matrix.
`max(C)` .... the maximum of each row.
`min(C)` .... the minimum in each row.
`sum(C)` .... sums each row.
`mean(C)` .... the average of each row.
`diag(C)` .... just the diagonal elements.
`inv(C)` .... inverse of the matrix.
`C'` .... transpose of the matrix.

Matrix decompositions:

`[L U P] = lu(C)`
`[Q R] = qr(C)`
`H = hess(C)` .... transform into a Hessian tri-diagonal matrix, which has the same eigenvalues as A.

Part III

Functions and Data