CONICS AS AXES AND JACOBIAN PROBLEM

By Shreeram S. Abhyankar

(0.1) CONICS AS AXES. In High-School Algebra we study factorization of polynomials. In College Analytic-Geometry we introduce the (X, Y)-axes to study geometric figures such as lines and conics. To put these subjects together we generalize the idea of axes thus.

DEFINITION. Polynomials $f(X, Y)$ and $g(X, Y)$ are said to form an **axes-pair** (or automorphic pair) if X and Y can be expressed as polynomials in f and g, i.e., we can find polynomials $u(X, Y)$ and $v(X, Y)$ such that

$$(^*) \qquad X = u(f(X, Y), g(X, Y)) \quad \text{and} \quad Y = v(f(X, Y), g(X, Y)).$$

We call $f(X, Y)$ an **axis** if f, g is an axes-pair for some g.

CONICS. Let $f(X, Y)$ be a nonconstant polynomial of degree $N > 0$ and consider the plane curve $C : f(X, Y) = 0$. If $N = 1$ then C is a line. If $N = 2$ then C is a conic and

$$f(X, Y) = aY^2 + bXY + cX^2 + pX + qY + r$$

where a, b, c, p, q, r are constants with either $a \neq 0$ or $b \neq 0$ or $c \neq 0$. A conic is either a circle or an ellipse or a hyperbola or a parabola or a pair of lines (which may or not be distinct).

FIRST EXERCISE. Show that a conic is an axis iff (= if and only if) it is a parabola.

HINT. The parabola $Y^2 - X$ is an axis because $Y^2 - X$ and Y form an axes-pair. If the conic f is not a parabola then it factors after adding a suitable constant; add 1 to the circle $X^2 + Y^2 - 1$, ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1$, hyperbola $\frac{X^2}{a^2} - \frac{Y^2}{b^2} - 1$ or $XY - 1$ (these standard forms are obtained after a suitable linear change of variables); if f is a pair of lines then take the constant to be zero. Therefore it only remains to show that (i) an axis must be irreducible (= not reducible = cannot be factored) and (ii) an axis remains an axis after adding a constant.

Assertion (ii) follows by noting that f, g axes-pair and a, b constants obviously implies $f + a, g + b$ axes-pair.

To prove assertion (i), for any $h(X, Y)$ let $h'(X, Y) = h(f(X, Y), g(X, Y))$ where f, g is an axes-pair. Then $(\bullet)$ $h \mapsto h'$ is a k- automorphism of $k[X, Y]$ where k is the field of coefficients, and hence h is irreducible iff h' is irreducible. Clearly $f = X'$ [i.e., taking $h(X, Y) = X$ we get $h'(X, Y) = f(X, Y)$] and hence f is irreducible because X is irreducible; thus f, g axis-pair implies f irreducible.

The abstract assertion $(\bullet)$ can be converted into an elementary argument thus. If (1) $h(X, Y) = h_1(X, Y)h_2(X, Y)$ then substituting f, g for X, Y on both sides we get (2) $h'(X, Y) = h'_1(X, Y)h'_2(X, Y)$ and conversely, in view of (*), substituting u, v for X, Y in both sides of (2) we get back (1).

DEFINITION. To generalize the First Exercise, for a polynomial $f(X, Y)$, by f^+ we denote the **degree form** of f, i.e., the highest degree terms in f; if $f = 0$ then we put $f^+ = 0$. f^+ gives the behaviour of f for large values of X, Y, and hence we call factors of f^+ **points at infinity** of f. In particular, we say that f has **only one point at infinity** to mean that f is a nonconstant irreducible polynomial such that $f^+ = a(bX+cY)^N$ where a, b, c are constants and N is the degree of f. Now we observe that a circle and an ellipse have two complex points at infinity, a hyperbola has two real points at infinity, and a parabola has only one point at infinity. This suggests the following generalization of the First Exercise.

SECOND EXERCISE. Show that f, g axes-pair implies f has only one point at infinity.

HINT. Every point of an irreducible projective variety is the center of a valuation of the function field of that variety, and conversely every valuation has a center. For details see [Ab3] and [Ab6].

FUTURE TASKS. To make further progress in understanding the concept of an axis, I suggest that you read my Kyoto Paper [Ab2]. You may disregard the first three pages of that paper, i.e., start with Chapter 2 on Decimal Expansion which is on pages 251-261. The Kyoto Paper is (except for the first 3 pages where it is summarized) completely self contained and can be understood by anybody with a high-school education. The other references at the end of the paper, in addition to some of my other books such as [Ab0] and [Ab7], contain a list of some classical algebra and analysis books which I highly recommend such as [Bôc], [Chr], [Edw], [Gib], [Gou], [Kap], [Phi], [SeR], [Wal] and [Zar].

(0.2) RESULTANT OR ALTERNATIVE HINT TO SECOND EXERCISE. As an alternative solution of the Second Exercise, in the classical vein, we can use the Resultant introduced by Sylvester in 1840 thus; see Chapter 9 (pages 374-404) of the Kyoto Paper [Ab2], or Lecture 30 (pages 267-273) of the Engineering Book [Ab3] or Lecture 4§1 (pages 100-104) of the Algebra Book I [Ab6].

Assuming n, m to be nonnegative integers, the Y-Resultant of two polynomials

$$\begin{aligned} f(Y) &= a_0 Y^n + a_1 Y^{n-1} + \cdots + a_n \\ g(Y) &= b_0 Y^m + b_1 Y^{m-1} + \cdots + b_m \end{aligned}$$

is the determinant

$$\mathrm{Res}_Y(f,g) = \det(\mathrm{Resmat}_Y(f,g))$$

of the $n+m$ by $n+m$ matrix

$$\mathrm{Resmat}_Y(f,g) = \begin{pmatrix}
a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \\
b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_m & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_m & 0 & \cdot & \cdot & \cdot & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & \cdot & \cdot & \cdot & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_m
\end{pmatrix}$$

where the first m rows consist of the coefficients of f and the last n rows consist of the coefficients of g. In detail, the first row starts with the coefficients of f, these are shifted one step to the right to get the second row, shifted two steps to the right to get the third row, and so on for the first m rows, then the $(m+1)$-st row starts with the coefficients of g, these are shifted one step to the right to get the $(m+2)$-nd row, and so on for the next n rows. The matrix is completed by stuffing zeroes elsewhere. The determinant $\mathrm{Res}_Y(f,g)$ is sometimes called the Sylvester resultant of f and g because it was introduced by Sylvester in his 1840 paper where he enunciated the following Basic Fact and Permuting, Isobaric, & Root Properties.

BASIC FACT (formula (34) on page 391 of Kyoto Paper). If the coefficients a_i, b_j belong to a domain R then we have: $\mathrm{Res}_Y(f,g) = 0 \Leftrightarrow n+m \neq 0$ and either $a_0 = 0 = b_0$ or f and g have a common root in some overfield of R.

PERMUTING PROPERTY (formula (9) on page 377 of Kyoto Paper). We have

$$\mathrm{Res}_Y(g,f) = (-1)^{mn}\mathrm{Res}_Y(f,g).$$

ISOBARIC PROPERTY (formula (14) on page 378 of Kyoto Paper). View the coefficients a_i, b_j as indeterminates over $\mathbb{Z}$. Give weight i to a_i, and j to b_j. Then $0 \neq \mathrm{Res}_Y(f,g) \in \mathbb{Z}[a_0,\dots,a_n,b_0,\dots,b_m]$ is isobaric of weight mn, i.e., for the weight of any monomial $a_0^{i_0}\dots a_n^{i_n}b_0^{j_0}\dots b_m^{j_m}$ occurring in $\mathrm{Res}_Y(f,g)$ we have $(\sum_{0\le r\le n} ri_r) + (\sum_{0\le s\le m} sj_s) = mn$. In particular, the principal diagonal $a_0^m b_m^n$ has weight mn, and it does not cancel out because there is no other term of b_m-degree n in the resultant; the principal diagonal of an $N \times N$ matrix (A_{ij}) is the term $A_{11}A_{22}\dots A_{NN}$. The resultant being isobaric of weight mn is the fundamental fact behind various cases of Bézout's Theorem.

ROOT PROPERTY (formula (28) on page 390 of Kyoto Paper). If the coefficients a_i, b_j belong to a domain R and $a_0 \neq 0$ then, upon writing

$$f(Y) = a_0 \prod_{1\le i\le n}(Y - \alpha_i)$$

with $\alpha_1,\dots,\alpha_n$ in an overfield of R, we have

$$\mathrm{Res}_Y(f,g) = a_0^m \prod_{1\le i\le n} g(\alpha_i).$$

By Root Prop we see that

$$\begin{cases} \text{if } a_0 \neq 0 \text{ and } g'(Y) = b'_0Y^m + b'_1Y^{m-1} + \dots + b'_m \text{ is such that} \\ g' = \nu g - hf \text{ for some constant } \nu \text{ and polynomial } h \\ \text{then } \mathrm{Res}_Y(f,g') = \nu^n \mathrm{Res}_Y(f,g). \end{cases} \tag{1}$$

Now if $m = n$ with $a_0 \neq 0 \neq a_n \neq 0 = a_1 = \dots = a_{n-1} = b_1 = \dots = b_{n-1}$ then

$$\begin{cases} \mathrm{Res}_Y(f,g) & \\ = a_n^{-n}\mathrm{Res}_Y(f, a_ng - b_nf) & \text{taking } (\nu,h) = (a_n,b_n) \text{ in (1)} \\ = a_n^{-n}\mathrm{Res}_Y(f,(a_nb_0 - b_na_0)Y^n) & \text{by simplifying} \\ = (-1)^{n^2}(a_nb_0 - b_na_0)^n & \text{by Permuting Prop and Root Prop} \\ & \text{if } a_nb_0 - b_na_0 \neq 0 \\ & \text{and obviously otherwise} \end{cases}$$

and hence

$$\begin{cases} \text{if } m = n \text{ with } a_0 \neq 0 = a_1 = \cdots = a_{n-1} = b_1 = \cdots = b_{n-1} \text{ then} \\ \text{(by above calculation in case } a_n \neq 0 \text{ and by Root Prop in case } a_n = 0) \\ \text{we get } \mathrm{Res}_Y(f, g) = (-1)^{n^2}(a_n b_0 - b_n a_0)^n. \end{cases}$$

Thus

$$\begin{cases} \text{if } m = n \text{ with } a_0 \neq 0 = a_1 = \cdots = a_{n-1} = b_1 = \cdots = b_{n-1} \\ \text{then } \mathrm{Res}_Y(f, g) = (-1)^{n^2}(a_n b_0 - b_n a_0)^n. \end{cases} \tag{2}$$

By (2) and the Isobaric Prop we get the following defo (= degree form) Property:

DEFO PROPERTY. In the set-up of the Isobaric Property, assume $m = n$, and consider the bivariate (= two variable) polynomial

$$\Psi(a_n, b_n) = \mathrm{Res}_Y(f, g) \in K[a_n, b_n]$$

over the field

$$K = \mathbb{Q}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}).$$

Then for the (total) (a_n, b_n)-degree and degree form we have

$$\deg(\Psi) = n \quad \text{with} \quad \Psi^+ = (-1)^{n^2}(a_n b_0 - b_n a_0)^n.$$

PROOF. By the Isobaric Property we see that $\deg(\Psi) \leq n$ and any term of degree n in Ψ must be devoid of $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$ and hence, by putting $a_1 = \cdots = a_{n-1} = b_1 = \cdots = b_{n-1} = 0$ in f and g, our assertion follows from (2).

COROLLARY. Assume that $m + n > 0$. Let $A_0, \ldots, A_n, B_0, \ldots, B_m$ be elements in a field k, let X, Y, Z be indeterminates over k, and let

$$P(Z) = A_0 Z^n + A_1 Z^{n-1} + \cdots + A_n$$
$$Q(Z) = B_0 Z^m + B_1 Z^{m-1} + \cdots + B_m$$

and

$$\Phi(X, Y) = \mathrm{Res}_Z(P(Z) - X, Q(Z) - Y) \in k[X, Y].$$

Then we have

$$\Phi(P(Z), Q(Z)) = 0. \tag{3}$$

Moreover, if $m = n$ with either $A_0 \neq 0$ or $B_0 \neq 0$ then

$$\deg(\Phi) = n \quad \text{and} \quad \Phi^+ = (-1)^{n^2}(A_0 Y - B_0 X)^n \tag{4}$$

and for any nonconstant irreducible $F(X, Y) \in k[X, Y]$ with $F(P(Z), Q(Z)) = 0$ we have

$$F^+ = \rlap{-}\theta\,(A_0 Y - B_0 X)^N \tag{5}$$

where $\deg(F) = N$ and $\rlap{-}\theta$ = **Abhyankar's nonzero** = a nonzero constant = an unspecified nonzero element of k.

PROOF. Taking an indeterminate T over $k(X, Y, Z)$ we obtain the equation $\mathrm{Res}_Z(P(Z) - P(T), Q(Z) - Q(T)) = \Phi(P(T), Q(T))$ and clearly $Z = T$ is a common solution of $P(Z) - P(T) = 0 = Q(Z) - Q(T)$, and hence by the Basic Fact we get $\Phi(P(T), Q(T)) = 0$ which yields (3). Taking $a_i = A_i$ and $b_i = B_i$ for $0 \leq i \leq n - 1$ with $a_n = A_n - X$ and $b_n = B_n - Y$, by the Defo Property we get (4). By (3) and (4) we get (5).

POLYNOMIAL CURVE. By a polynomial curve over a field k we mean a nonconstant irreducible $f(X,Y) \in k[X,Y]$ for which there exist $P(Z), Q(Z)$ in $k[Z]$, at least one of which is not in k, such that $f(P(Z), Q(Z)) = 0$. Taking $F(X,Y) = f(X,Y)$ and putting $m = n = \max(\deg_Z P(Z), \deg_Z Q(Z))$ in the above Corollary we see that a polynomial curve has at most one point at infinity.

(0.3) EQUATION SOLVING. The genesis of resultants is the other topic studied in high-school algebra which consists of the various solvings of polynomial equations. First linear equations are discussed culminating in Cramer's Rule for solving m equations in n variables. Then Bhaskaracharya's 1150 A.D. verse of Shreedharachaya's 500 A.D. completing the square method of solving one-variable quadratic equations is studied. Turning to simultaneously solving two one-variable equations $f(Y) = 0$ and $g(Y) = 0$, or rather to finding a condition for them to have a common solution, we get their resultant $\mathrm{Res}_Y(f,g)$ as depicted in the beginning of (0.2).

As reference for the above paragraph see pages 1-2, 100-104, 172-188 of my 2006 Algebra Book I [Ab6], and for a classical treatment of the matter see the 1907 Higher Algebra Book of Bôcher [Bôc].

In view of the description of Polynomial Curves given at the end of (0.2), to complete the explanation of the Second Exercise, it only remains to show that if f, g is an axes-pair then f is a polynomial curve. This can be done either abstractly or concretely thus.

STRICT POLYNOMIAL CURVE. Abstractly, for the polynomial ring $R = k[X,Y]$, as in ($\bullet$) of the First Exercise, $h \mapsto h'$ gives a k-homomorphism $H : R \to R$. Likewise, for any $w(X,Y)$ let $w^*(X,Y) = w(u(X,Y), v(X,Y))$. Then we get the k-homomorphism $W : R \to R$ given by $w \mapsto w^*$. The display (*) at the beginning of (0.1) yields $HW = WH =$ the identity map $I : R \to R$, and hence H and W are automorphisms. Consequently for the residue class epimorphism $D : R \to R/(fR)$ we have $D(R) = k[D(g)]$ and hence there is a unique k-epimorphism $E : R \to k[Z]$ whose kernel is fR such that $E(g) = Z$. Let $P(Z) = E(X)$ and $Q(Z) = E(Y)$. Then $f(P(Z), Q(Z)) = 0$ and either $P(Z) \notin k$ or $Q(Z) \notin k$. Thus f is a polynomial curve. Actually f is a **strict polynomial curve**, i.e., $(P(Z), Q(Z))$ is an **epimorphic pair** which means that

$$Z = G(P(Z), Q(Z))$$

for some $G(X,Y) \in k[X,Y]$; indeed $G(X,Y) = g(X,Y)$.

Concretely speaking, putting $P(Z) = u(0,Z)$ and $Q(Z) = v(0,Z)$ gives us

$$f(P(Z), Q(Z)) = 0 \quad \text{and} \quad g(P(Z), Q(Z)) = Z.$$

EPIMORPHIC PAIR. In the Kyoto Paper it is proved that, in case of a characteristic zero field k, we have: $(P(Z), Q(Z))$ epimorphic pair implies either the degree of $P(Z)$ divides the degree of $Q(Z)$ or the degree of $Q(Z)$ divides the degree of $P(Z)$, and from this it is deduced that every strict polynomial curve is an axis. Examples show that both these are false for nonzero characteristic.

(0.4) NEWTON'S THEOREM AND JACOBIAN PROBLEM. The above degree dividing property of an epimorphic pair gives us entry into the Jacobian Problem which is the high-school incarnation of the inverse function theorem of calculus. For a detailed treatment of the inverse function theorem of calculus together with its mate the implicit function theorem, see Chapter II of my Local Analytic Book [Ab0].

The said implicit function theorem is the often neglected foundation of the method of implicit differentiation as exemplified by the following example.

$$f(X,Y)=0 \Rightarrow f_X dX + f_Y dY = 0 \Rightarrow \frac{dY}{dX}=\frac{-f_X}{f_Y}.$$

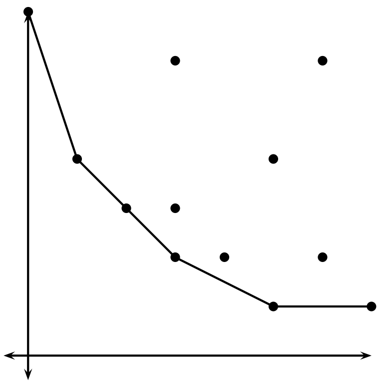
But what if $f_X = f_Y = 0$ such as when $f(X,Y) = Y^2 - X^3$ (cusp) or $f(X,Y) = Y^2 - X^2 - X^3$ (node) and we are evaluating $\frac{dY}{dX}$ at the origin $(X,Y) = (0,0)$? For the resulting theory of singularities see Lectures 1 and 5 of my Engineering Book [Ab3].

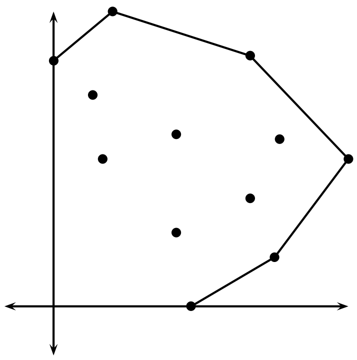
At any rate, the inverse function theorem of calculus says that if n functions of n variables (with sufficiently many continuous partial derivatives) are zero at the origin but their jacobian is not then, locally near the origin, the variables are functions of the functions. The Jacobian Problem asks if this remains true if the only permissible functions are polynomials. For instance, in case of $n = 2$, given polynomials $f(X,Y), g(X,Y)$ with $J(f,g) = \not{0}$ we are asking if X, Y are polynomials in f, g, i.e., if f, g is an axes-pair. Here $J(f,g) = f_X g_Y - g_X f_Y$ with subscripts denoting partial derivatives. Surprisingly, the answer is not known even for $n = 2$. More about this later on. At present see Lectures 22-23 of the Engineering Book as well as my recent papers [Ab5] and [Ab8]. At any rate, according to these references, the 2 variable Jacobian Problem is equivalent to showing that if $J(f,g) = \not{0}$ then out of the (total) degrees of f and g, one divides the other.

To prove the degree dividing property of an epimorphic pair, we apply Newton's Theorem on fractional expansion to the minimal equation $\Phi(X,Y) = 0$ satisfied by the pair. For details see [Ab2].

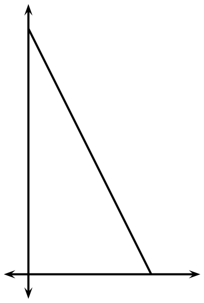
(0.5) DEGREEWISE NEWTON POLYGON. As an aid to proving his theorem, Newton considered the support $\mathrm{Supp}(\Phi)$ = the set of all integer pairs (i,j) such that the coefficient of X^iY^j in $\Phi(X,Y)$ is nonzero, and he drew the polygon which is the boundary of the convex hull of that support, as depicted in the left hand picture below. Good discussions of the Newton Polygon can be found on pages 373-396 of volume II of Chrystal's book [Chr] and on pages 98-106 of Walker's book [Wal]. One important property of the Newton Polygon is the fact that the slopes of the various sides are the orders of the various roots of Φ. Moreover, the integral part of the slope of the first side equals the number of Quadratic Transforms required to decrease multiplicity of the plane analytic curve $\Phi(X,Y) = 0$. See the books [Ab3] and [Ab6].

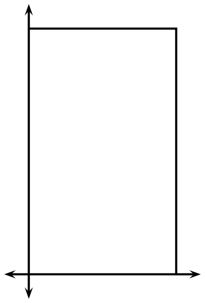
Now assume that f is a nonzero polynomial in X and Y. Then the support is a nonempty finite set. So the convex hull will look something like the following diagram on the right, and we call it the Degreewise Newton Polygon or DNP. Since this was introduced in my lecture notes [Ab1] and [Ab4], people also call it the Abhyankar Polygon.



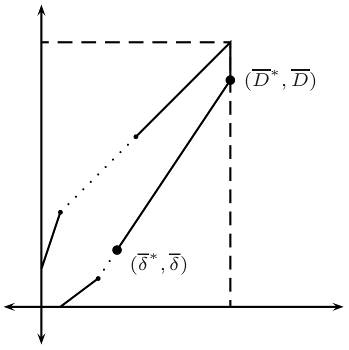


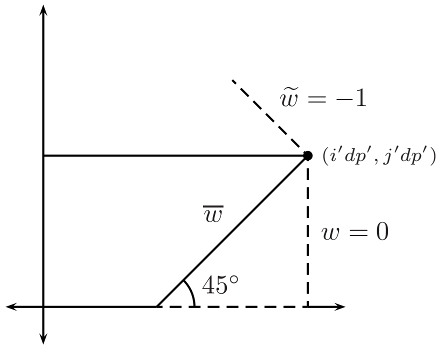
(0.6) TRIANGLE AND FULL RECTANGLE. The significance of the DNP for the Jacobian Problem is the fact that a positive answer to that problem is equivalent to showing that if $J(f,g) = \cancel{0}$ then the DNP of f is a triangle as depicted in the following picture on the left, while a negative answer is equivalent to showing that for some pair f, g with $J(f,g) = \cancel{0}$ the DNP of f is a full rectangle as depicted in the following picture on the right.





(0.7) SQUEEZED RECTANGLE AND SHORTSQUEEZED RECTANGLE. If $J(f,g) = \cancel{0}$ but f has two points at infinity then, by a change of coordinates, its DNP can be arranged to be squeezed vertical rectangle as depicted in the following picture on the left, or a shortsqueezed horizontal rectangle as depicted in the following picture on the right where the exhibited angle could be smaller than 45°.





Epilogue

MANGALACHARAN
ATA VISHVATMAKE DEVE | YENE VAGYADNYE TOSHAVE
TOSHONI MAJA DYAVE | PASAYDANA HE
GANITAVIDYECHEE JAGRUTEE | KARONIYA SARVA JAGATEE
PRADNYASURYE UJALATEE | SUKHAVAYA SAKALA JANA

Here is a free Paraphrase of the above MAGALACHARAN = INVOCATION in my mother tongue MARATHI whose founding father DNYANESHVAR composed the first two lines around 1250 A.D. to which I added the last two lines.

PARAPHRASE. May the Lord God of the Universe be pleased with my recounting of the story of algebra and geometry which are the essence of our beloved subject of mathematics. Being pleased may he shower his blessings upon us and make our endeavor pleasurable.

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