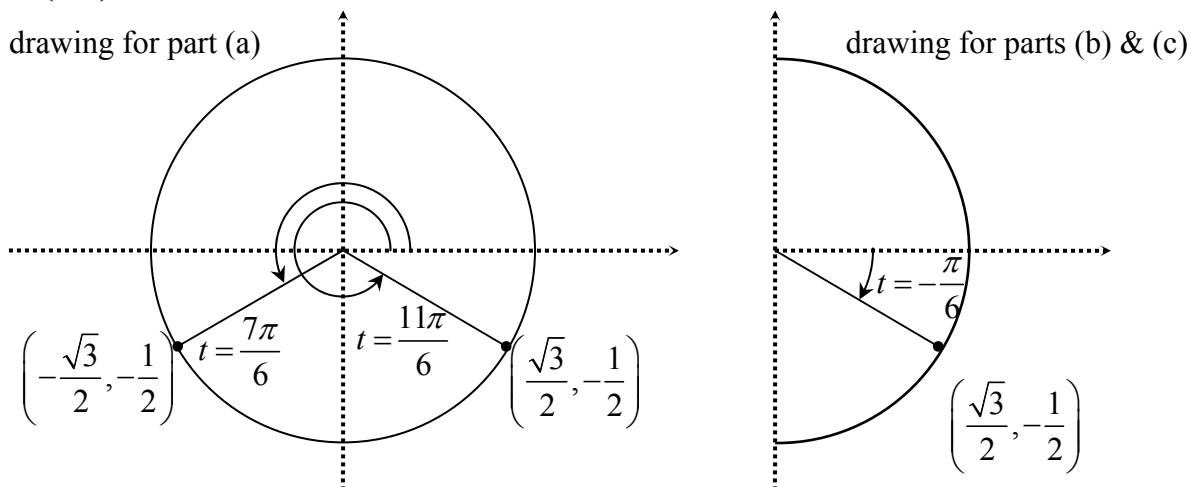


1 (a) Solution: $t = \frac{7\pi}{6}$ and $t = \frac{11\pi}{6}$. See drawing below.

(b) Solution: $t = -\frac{\pi}{6}$. See drawing below.

(c) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. (Observe that questions (b) & (c) are actually the same question.)



2 (a) $y = 3 \cos(2t)$. To see why, make the following sequence of graphs:

1st graph: $y = \cos(t)$ some key points: $\left(-\frac{\pi}{2}, 0\right), (0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1)$

2nd graph: $y = \cos(2t)$ some key points: $\left(-\frac{\pi}{4}, 0\right), (0, 1), \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, -1\right)$

3rd graph: $y = 3 \cos(2t)$ some key points: $\left(-\frac{\pi}{4}, 0\right), (0, 3), \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, -3\right)$

(b) $y = 3 \sin\left(2t + \frac{\pi}{2}\right)$. To see why, make the following sequence of graphs:

1st graph: $y = \sin(t)$ some key points: $(0, 0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right)$

2nd graph: $y = \sin\left(t + \frac{\pi}{2}\right)$ some key points: $\left(-\frac{\pi}{2}, 0\right), (0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1)$

3rd graph: $y = \sin\left(2t + \frac{\pi}{2}\right)$ some key points: $\left(-\frac{\pi}{4}, 0\right), (0, 1), \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, -1\right)$

4th graph: $y = \sin\left(2t + \frac{\pi}{2}\right)$ some key points: $\left(-\frac{\pi}{4}, 0\right), (0, 3), \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, -3\right)$

3 I will discuss the properties of the four graphs, but will leave the drawing to you.

(a) 1st graph: $r = \tan(t)$. The graph has these properties:

input: t

output: r

horizontal axis (the “input axis”): t

$$\text{domain: } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

vertical axis (the “output axis”): r

$$\text{range: } r \in \mathbb{R}$$

$$3 \text{ key points: } \left(-\frac{\pi}{4}, -1\right), (0, 0), \left(\frac{\pi}{4}, 1\right)$$

$$\text{vertical asymptotes at } t = -\frac{\pi}{2} \text{ and } t = \frac{\pi}{2}.$$

(b) 2nd graph: $t = \arctan(r)$. Since this is the inverse function for $r = \tan(t)$, we obtain the new graph by flipping the 1st graph across the 45° line. We will have to re-label the coordinate axes, flip the key points, and flip the asymptotes. (The asymptotes will now be horizontal.) The new graph has these properties:

input: r

output: t

horizontal axis (the “input axis”): r

$$\text{domain: } r \in \mathbb{R}$$

vertical axis (the “output axis”): t

$$\text{range: } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$3 \text{ key points: } \left(-1, -\frac{\pi}{4}\right), (0, 0), \left(1, \frac{\pi}{4}\right)$$

$$\text{horizontal asymptotes at } t = -\frac{\pi}{2} \text{ and } t = \frac{\pi}{2}.$$

(c) 3rd graph: $t = \arctan(r-1)$. This transformation of the 2nd graph is a translation of 1 unit in the positive r direction. That means that we will make this 3rd graph by moving the 2nd graph one unit to the right. We will have to add one to the x -values of the three key points. The new graph has these properties:

input: r

output: t

horizontal axis (the “input axis”): r

$$\text{domain: } r \in \mathbb{R}$$

vertical axis (the “output axis”): t

$$\text{range: } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$3 \text{ key points: } \left(0, -\frac{\pi}{4}\right), (1, 0), \left(2, \frac{\pi}{4}\right)$$

$$\text{horizontal asymptotes at } t = -\frac{\pi}{2} \text{ and } t = \frac{\pi}{2}.$$

(d) 4th graph: $t = \arctan(2r-1)$. This is transformation by “horizontal division”. To make this 4th graph, we will have to divide the x -values of all of the points on the 3rd graph by 2. In particular, we will have to divide the x -values of the three key points by 2. The new graph has these properties:

input: r

output: t

horizontal axis (the “input axis”): r

$$\text{domain: } r \in \mathbb{R}$$

vertical axis (the “output axis”): t

$$\text{range: } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$3 \text{ key points: } \left(0, -\frac{\pi}{4}\right), \left(\frac{1}{2}, 0\right), \left(1, \frac{\pi}{4}\right)$$

$$\text{horizontal asymptotes at } t = -\frac{\pi}{2} \text{ and } t = \frac{\pi}{2}.$$

4] As in the previous problem, I will give the properties of each of the graphs. You make the graphs.

(a) 1st graph: $y = 5^{(x)}$. The graph has these properties:

input: x

output: y

horizontal axis (the “input axis”): x

domain: $x \in \mathbb{R}$

vertical axis (the “output axis”): y

range: all $y > 0$

$$3 \text{ key points: } \left(-1, \frac{1}{5}\right), (0, 1), (1, 5)$$

horizontal asymptote at $y = 0$.

(b) 2nd graph: $x = \log_5(y)$. Since this is the inverse function for $y = 5^{(x)}$, we obtain the new graph by flipping the 1st graph across the 45° line. We will have to re-label the coordinate axes, flip the key points, and flip the asymptotes. (The asymptotes will now be horizontal.) The new graph has these properties:

input: y

output: x

horizontal axis (the “input axis”): y

domain: all $y > 0$

vertical axis (the “output axis”): x

range: $x \in \mathbb{R}$

$$3 \text{ key points: } \left(\frac{1}{5}, -1\right), (1, 0), (5, 1)$$

vertical asymptote at $y = 0$.

(c) 3rd graph: $x = 2 \log_5(y)$. This transformation multiplies all of the outputs by the number 2. That means that we will get the 3rd graph by multiplying the right coordinate of each point on the 2nd graph by 2. The new graph has these properties:

input: y

output: x

horizontal axis (the “input axis”): y

domain: all $y > 0$

vertical axis (the “output axis”): x

range: $x \in \mathbb{R}$

$$3 \text{ key points: } \left(\frac{1}{5}, -2\right), (1, 0), (5, 2)$$

vertical asymptote at $y = 0$.

(d) 4th graph: $x = 2 \log_5(y) - 2$. This transformation subtracts 2 from all of the outputs. That means that we will get the 4th graph by subtracting 2 from the right coordinate of each point on the 3rd graph. The graph has these properties:

input: y

output: x

horizontal axis (the “input axis”): y

domain: all $y > 0$

vertical axis (the “output axis”): x

range: $x \in \mathbb{R}$

3 key points: $\left(\frac{1}{5}, -4\right), (1, -2), (5, 0)$

vertical asymptote at $y = 0$.

$$\boxed{5} \text{ (a) } Q(t) = 50 \left(\frac{1}{2}\right)^{\frac{t}{578}}$$

$$\text{(b) } Q(100) = 50 \left(\frac{1}{2}\right)^{\frac{100}{578}}$$

$$\text{(c) } 10 = Q(t) = 50 \left(\frac{1}{2}\right)^{\frac{t}{578}} \quad \text{We must solve for } t. \text{ We start by dividing both sides by 50.}$$

$$\frac{1}{5} = \left(\frac{1}{2}\right)^{\frac{t}{578}}$$

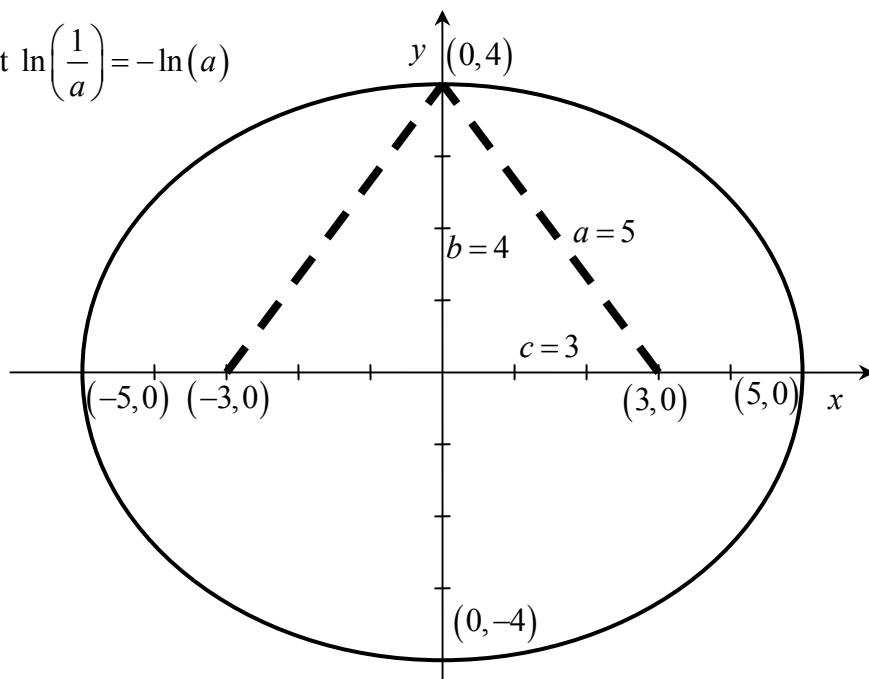
$$\ln\left(\frac{1}{5}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{578}}\right) \quad \text{here we just took the logarithm of both sides}$$

$$= \frac{t}{578} \ln\left(\frac{1}{2}\right) \quad \text{using fact that } \ln(a^b) = b \ln(a)$$

$$-\ln(5) = -\frac{t}{578} \ln(2) \quad \text{using fact that } \ln\left(\frac{1}{a}\right) = -\ln(a)$$

$$t = 578 \frac{\ln(5)}{\ln(2)}$$

$\boxed{6}$ The general form of the equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The fact that the point $(5, 0)$ is on the ellipse tells us that $a = 5$; because $(0, 4)$ is on the ellipse, we know that $b = 4$. Because the major axis of the ellipse is on the x -axis, we know that the two foci must also be on the x -



axis. Their coordinates will be $(c, 0)$ and $(-c, 0)$. We must find c .

As we discussed in class, the length L of string needed to construct the ellipse is equal to the length of the major axis. In this case, that means that $L = 2a$. We can imagine pulling the string in the y -direction, away from the foci. The resulting configuration will look as shown in the picture above. Because the bent string (the heavy dotted line) has total length $L = 2a$, each half of the bent string will have length a . So the hypotenuse of the little triangle on the right will be 5 units long. We will be able to pull the string a distance $b = 4$ in the y -direction because it has to reach just to the edge of the ellipse, which is at the point $(0, 4)$. Therefore, the height of the little triangle on the right will be 4 units. From the picture, we see that $c^2 + b^2 = a^2$. Plugging in $a = 5$ and $b = 4$, we find that $c = 3$.

□ Because we are told that there are foci at $(0, 5)$ and $(0, -5)$, and vertices at $(0, 4)$ and $(0, -4)$, we know that the hyperbola must open up and down. The final picture will look something like the one to the right. This picture is not finished, because it does not give equations for the asymptotes. For that, we will need to know more about the equation for the hyperbola.

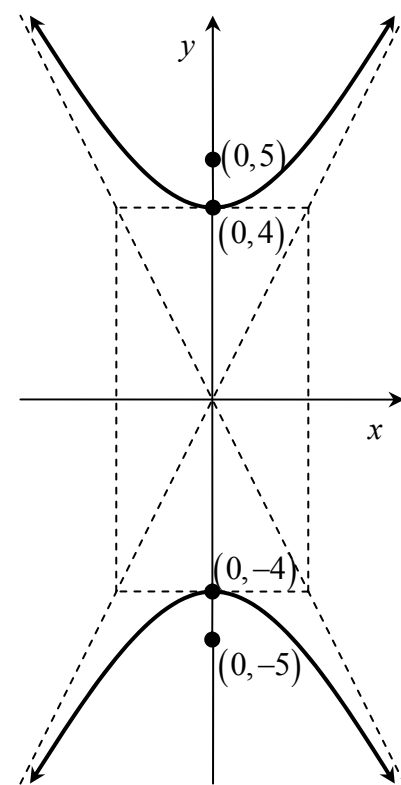
We know that the equation must of one of the following two forms:

$$\text{form 1: } \frac{x^2}{(\text{something})^2} - \frac{y^2}{(\text{something else})^2} = 1$$

In this case, there will be points of type $(x, 0)$, but there will not be any points of type $(0, y)$.

$$\text{form 2: } \frac{y^2}{(\text{something})^2} - \frac{x^2}{(\text{something else})^2} = 1.$$

In this case, there will be points of type $(0, y)$, but there will not be any points of type $(x, 0)$.



Because our hyperbola has a point $(0, 4)$ but does not have any points of type $(x, 0)$, our hyperbola must be described by an equation of form 2. The fact that the point $(0, 4)$ is on the graph tells us that

$$\frac{4^2}{(\text{something})^2} - \frac{0^2}{(\text{something else})^2} = 1$$

$$\frac{4^2}{(\text{something})^2} = 1$$

$$4^2 = (\text{something})^2$$

With this information, we can update our equation. We now know that it will be of the form

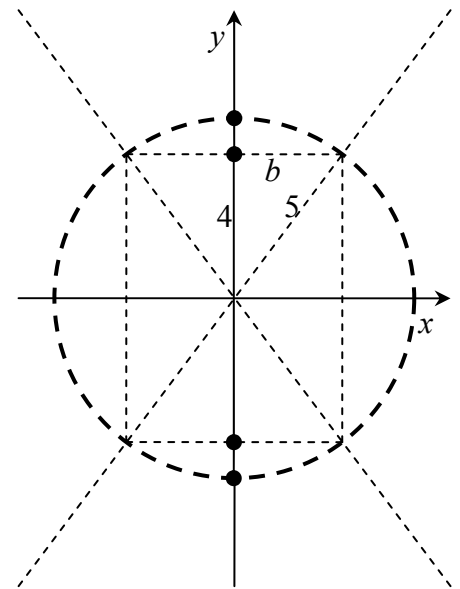
$$\frac{y^2}{4^2} - \frac{x^2}{(\text{something else})^2} = 1.$$

However, we should be cautious about assigning letter names to the “something else”, because we know that there is always some confusion about that. We have seen both $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ used in the book and in class, so we should be careful about our choice of where the “ a ” and “ b ” go.

The problem statement for part (a) says “For this problem, $c = 5$ and $a = 4$.” That means that the writer (me) has already made a choice of letter assignments. Apparently, I’ve decided that the number 4 will be called “ a ”. That means that the thing that is sitting under the “ y^2 ” in the equation must be an “ a^2 ”. Therefore, the thing that is sitting under the “ x^2 ” in the equation must be a “ b^2 ”. So, our equation must be of the form

$$\frac{y^2}{4^2} - \frac{x^2}{b^2} = 1.$$

In class and in the book, we have seen that a picture like the one shown to the right can be used to find b . From this picture, we can see that $b^2 + 4^2 = 5^2$. Therefore, $b = 3$. Thus, the equation that describes the hyperbola must be $\frac{y^2}{4^2} - \frac{x^2}{3^2} = 1$. From the picture, we see that the asymptotes are described by the equations $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$.



With that information, we can make the final sketch:

