

# Roots, Linear Factors, and Sign Charts

review of background material for Math 163A (Barsamian)

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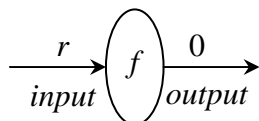
## 1. Introduction

The “sign chart” is a tool for determining when a function is positive, negative, or zero. It uses concepts from algebra and precalculus. If you took either of those courses at the college level, you will have encountered the sign chart there, although it may have had a different name, or no name at all. If you took those courses only in high school, then you might not have learned about sign charts. These notes are meant to be complete enough to give a student who has never seen sign charts an understanding sufficient for the needs of Math 163A. We begin with discussions of “roots” and “linear factors”, concepts that are crucial to understanding how sign charts work.

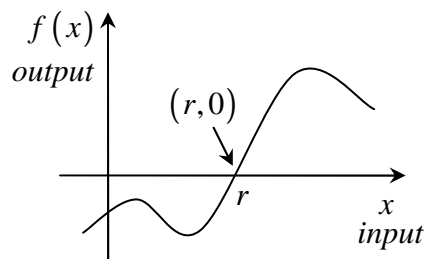
## 2. Roots

definition: *root*

- words: “ $r$  is a root of  $f$ .”
- usage:  $f$  is a function and  $r$  is a number.
- meaning:  $f(r) = 0$
- meaning, in words: If  $r$  is used as input to the function  $f$ , then the resulting output is zero.
- machine diagram:



- Graph:



- Observation: If  $r$  is a root of  $f$ , then the point  $(r, 0)$  is on the graph of  $f$ , and vice-versa.

## Examples

- 1) The function  $f(x) = x^2 - 5x + 6$  has roots  $r = 2$  and  $r = 3$ , because

$$f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$$

$$f(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$$

The graph of  $f$  is a parabola facing up, with  $x$ -intercepts  $(2, 0)$  and  $(3, 0)$ .

- 2) The function  $f(x) = x^2 - 5$  has roots  $r = \sqrt{5}$  and  $r = -\sqrt{5}$ , because

$$f(\sqrt{5}) = (\sqrt{5})^2 - 5 = 5 - 5 = 0$$

$$f(-\sqrt{5}) = (-\sqrt{5})^2 - 5 = 5 - 5 = 0$$

The graph of  $f$  is a parabola facing up, with  $x$ -intercepts  $(\sqrt{5}, 0)$  and  $(-\sqrt{5}, 0)$  and  $y$ -intercept  $(0, -5)$ .

- 3) The graph of the function  $f(x) = x^2 + 5$  is a standard parabola facing up. It has a  $y$ -intercept  $(0, 5)$ , and has no  $x$ -intercepts. That tells us that the function  $f(x) = x^2 + 5$  has no roots. There is no real number  $x$  such that  $x^2 + 5 = 0$ .

- 4) The function  $f(x) = x^2 + 2x + 4$  can be rewritten  $f(x) = (x+1)^2 + 3$ . From this, we can tell that the graph of  $f$  is a standard parabola facing up, moved one unit to the left and three units up. It will have its vertex at the point  $(-1, 3)$ , its  $y$ -intercept at  $(0, 4)$ , and will have no  $x$ -intercepts. The fact that the graph has no  $x$ -intercepts tells us that the function  $f(x) = x^2 + 2x + 4$  has no roots. There is no real number  $x$  such that  $x^2 + 2x + 4 = 0$ .

- 5) The function  $f(x) = x^3 - 2x^2 - x + 2$  has roots  $r = -1$ ,  $r = 1$ , and  $r = 2$ , because

$$f(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2 = -1 - 2 + 1 + 2 = 0$$

$$f(1) = (1)^3 - 2(1)^2 - (1) + 2 = 1 - 2 - 1 + 2 = 0$$

$$f(2) = (2)^3 - 2(2)^2 - (2) + 2 = 8 - 8 - 2 + 2 = 0$$

The graph of  $f$  has  $x$ -intercepts  $(-1, 0)$ ,  $(1, 0)$ , and  $(2, 0)$ .

- 6) The function  $f(x) = x^3 - 4x^2 + 5x - 2$  has roots  $r = 1$ , and  $r = 2$ , because

$$f(1) = (1)^3 - 4(1)^2 + 5(1) - 2 = 1 - 4 + 5 - 2 = 0$$

$$f(2) = (2)^3 - 4(2)^2 + 5(2) - 2 = 8 - 16 + 10 - 2 = 0$$

It turns out that there are no other roots. The graph of  $f$  has  $x$ -intercepts  $(1, 0)$  and  $(2, 0)$ .

- 7) The graph of the function  $f(x) = x^3 - 8$  looks like a standard  $y = x^3$  graph but moved down 8 units. It has  $x$ -intercept  $(2, 0)$ . Therefore, we know that the function  $f(x) = x^3 - 8$  must have only one root,  $r = 2$ . We can confirm that this is a root by computing  $f(2) = (2)^3 - 8 = 8 - 8 = 0$ .

Notice that in each of the examples above, the number of roots of  $f$  was less than or equal to the degree of  $f$ . This is a general fact about polynomials that we will discuss later. The first four examples above dealt with quadratic functions (polynomials of degree 2). One can find roots for these by factoring or by using the quadratic formula. We will discuss factoring later. The last three examples dealt with cubic functions (polynomials of degree 3). Finding roots of polynomials of degree 3 or higher can be very difficult, and there is a certain amount of guesswork involved. There are techniques to help one make smarter guesses, but these techniques will not be discussed in our course. One can attempt to factor the polynomial, but that will also involve guesswork. In upcoming sections we will see that in fact, finding the roots of a polynomial is equivalent to finding the factors of the polynomial. So, it should be no surprise that if finding roots is difficult, then finding factors will also be difficult. In Math 163A, most problems that involve finding roots (or finding factors) only involve quadratic polynomials. In most problems involving polynomials of degree 3 or higher, you will be given the roots (or given the factors).

### 3. Linear Factors

Linear functions are functions of the form  $y = mx + b$ ; their graphs are straight lines. A related concept is the “linear factor”:

definition: *linear factor*

- 1) words: “linear factor with root  $r$ ”
- 2) usage:  $r$  is a some real number
- 3) meaning: the expression  $(x - r)$

Examples

- 1) The polynomial  $f(x) = x^3 - 2x^2 - x + 2$  can be written as a product of linear factors,  $f(x) = (x + 1)(x - 1)(x - 2)$ .
- 2) The polynomial  $f(x) = x^3 - 4x^2 + 5x - 2$  can be written as a product of linear factors,  $f(x) = (x - 1)^2(x - 2) = (x - 1)(x - 1)(x - 2)$ .
- 3) The polynomial  $f(x) = x^3 - 8$  can be factored as  $f(x) = (x^2 + 2x + 4)(x - 2)$ , but only one of these factors is a linear factor. The other factor,  $(x^2 + 2x + 4)$  cannot be broken down further into linear factors.

There is an important correspondence between the roots of a polynomial and its linear factors:

$$\text{roots of } f \quad \Leftrightarrow \quad \text{factors of } f$$

$$r \text{ is a root of } f \iff (x-r) \text{ is a factor of } f$$

In Math 163A, we will not discuss the proof that this correspondence holds. We will just take it as a fact. For examples illustrating this correspondence, we will revisit the 7) examples from Section 2, above.

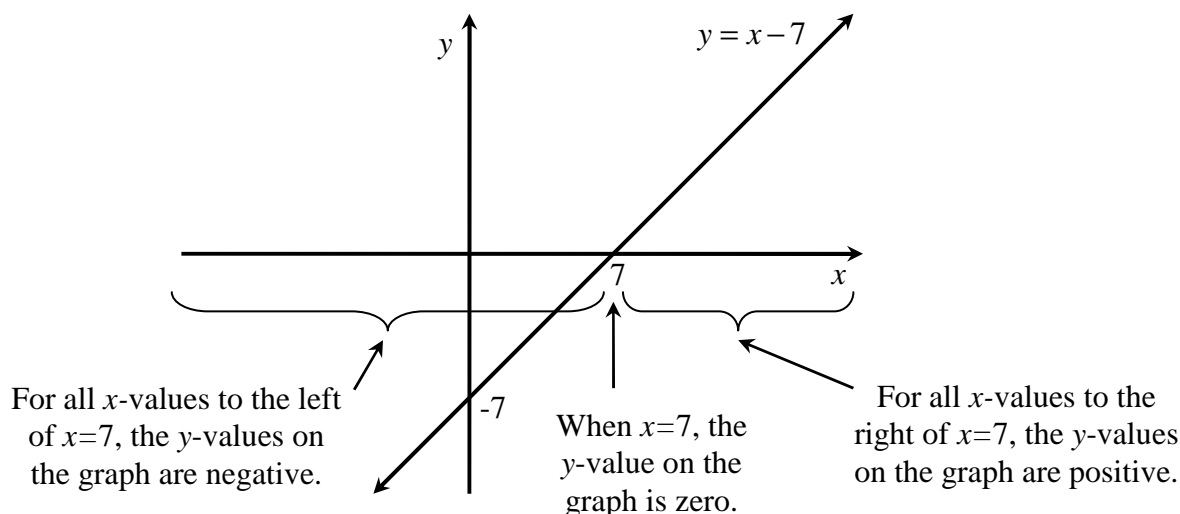
- 1) The function  $f(x) = x^2 - 5x + 6$  has roots  $r = 2$  and  $r = 3$  that correspond to the linear factors in the factorization  $f(x) = (x-2)(x-3)$ .
- 2) The function  $f(x) = x^2 - 5$  has roots  $r = \sqrt{5}$  and  $r = -\sqrt{5}$  that correspond to the linear factors in the factorization  $f(x) = (x-\sqrt{5})(x+\sqrt{5})$ .
- 3) The function  $f(x) = x^2 + 5$  has no roots, and  $f$  cannot be factored into the form  $f(x) = (x-a)(x-b)$ .
- 4) The function  $f(x) = x^2 + 2x + 4$  has no roots, and  $f$  cannot be factored into the form  $f(x) = (x-a)(x-b)$ .
- 5) The function  $f(x) = x^3 - 2x^2 - x + 2$  has roots  $r = -1$ ,  $r = 1$ , and  $r = 2$  that correspond to the linear factors in the factorization  $f(x) = (x+1)(x-1)(x-2)$ .
- 6) The function  $f(x) = x^3 - 4x^2 + 5x - 2$  has roots  $r = 1$ , and  $r = 2$  that correspond to the linear factors in the factorization  $f(x) = (x-1)^2(x-2) = (x-1)(x-1)(x-2)$ .
- 7) The function  $f(x) = x^3 - 8$  has only one root,  $r = 2$ . We have seen that  $f$  can be factored  $f(x) = (x^2 + 2x + 4)(x-2)$ , but  $f$  cannot be broken down completely into linear factors. The linear factor  $(x-2)$  corresponds to the root  $r = 2$ , but the factor  $(x^2 + 2x + 4)$  is not a linear factor and does not correspond to any roots.

The correspondence between roots and linear factors is the reason that finding roots is equivalent to finding linear factors. Furthermore, the above examples illustrate why the number of roots of a polynomial is always less than or equal to the degree of the polynomial. Indeed, a polynomial can never have more roots than linear factors. But example 6) illustrates that many linear factors can correspond to the same root, and example 7) illustrates that there may be other factors that are not linear factors and do not correspond to any roots. When all of the factors are multiplied together, the resulting highest power of  $x$  may be greater than the number of roots—that is, the degree may be greater than the number of roots—but it can never be less.

### 4. Sign Charts

The sign chart method exploits a simple-but-powerful observation: It is easy to determine when a single linear factor is positive, negative, or zero, and so it should also be easy to determine when a product of a bunch of linear factors is positive, negative or zero. We start by considering a single linear factor.

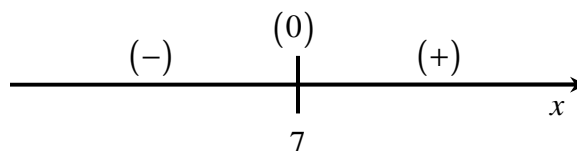
Consider the linear function  $y = x - 7$  and its graph.



Observe that the graph has a  $y$ -intercept at  $(0, -7)$  and an  $x$ -intercept at  $(7, 0)$ . Furthermore, note the following three important properties:

- For all  $x$ -values to the left of  $x=7$ , the  $y$ -values on the graph are negative.
- When  $x=7$ , the  $y$ -value on the graph is zero.
- For all  $x$ -values to the right of  $x=7$ , the  $y$ -values on the graph are positive.

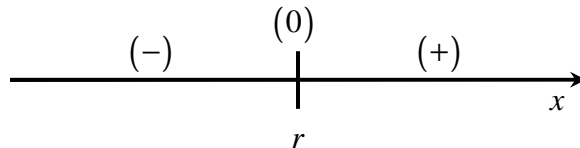
These three properties can be illustrated by putting  $+$ ,  $-$ , and  $0$  symbols on a number line. We will call the resulting figure a “sign chart” for the function  $y = x - 7$ .



sign chart for the function  $y = x - 7$

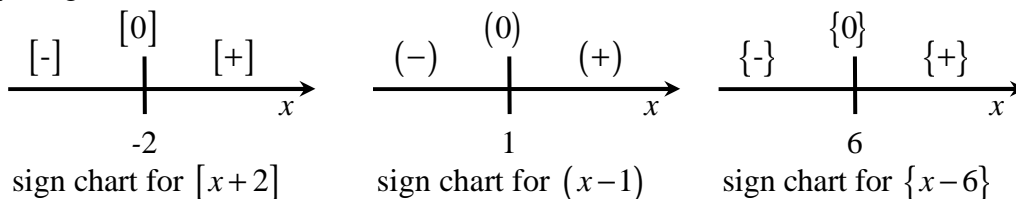
Observe that in the above discussion, there is nothing special about the number 7. Indeed, we could simply replace every “7” with the symbol “ $r$ ”, and the resulting words and pictures would all be equally valid. (Except for one necessary change: We would need to omit the  $y$ -axis from the graph of  $y = x - r$ , because without knowing the value of  $r$ , we would have no way of knowing whether the  $y$ -axis belongs to the left or to the right of the position where  $x = r$ . However, notice that the  $y$ -axis played no role in the sign chart, so we would not have to change the sign chart at all.) Furthermore, we can narrow our focus: instead of considering the sign behavior of function  $y = x - r$ , we can consider the sign

behavior of the linear factor  $(x-r)$ . The reason for this is that in the future, we will build more complicated functions from many linear factors, and we will want to consider the sign behavior of each linear factor independently. Summarizing this paragraph, we can now present the following sign chart for the linear factor  $(x-r)$ .

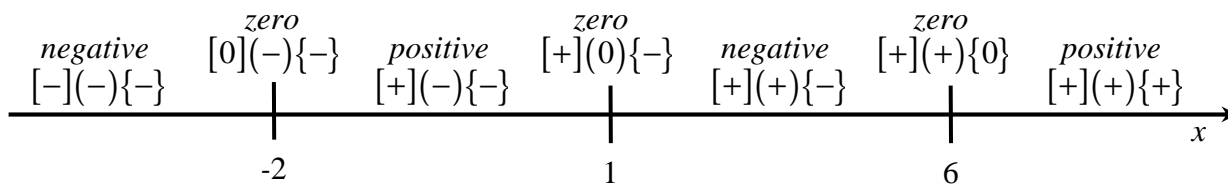


sign chart for the linear factor  $(x-r)$

Now consider the polynomial function  $f(x) = x^3 - 5x^2 - 8x + 12 = (x+2)(x-1)(x-6)$ . It is a product of three linear factors. For clarity, let's give each linear factor a distinct type of brackets, writing the function as  $f(x) = [x+2](x-1)\{x-6\}$ . The sign behavior of each of the linear factors can be illustrated by a sign chart.

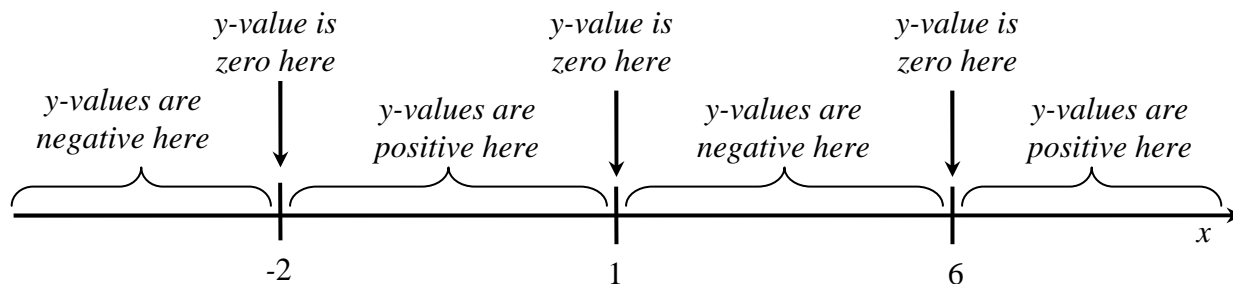


But we can also consider the sign behavior of the *product* of the three linear factors. Each linear factor's sign behavior is still dictated by its own sign chart, so in a sense, we have to combine the three sign charts above into a single chart.



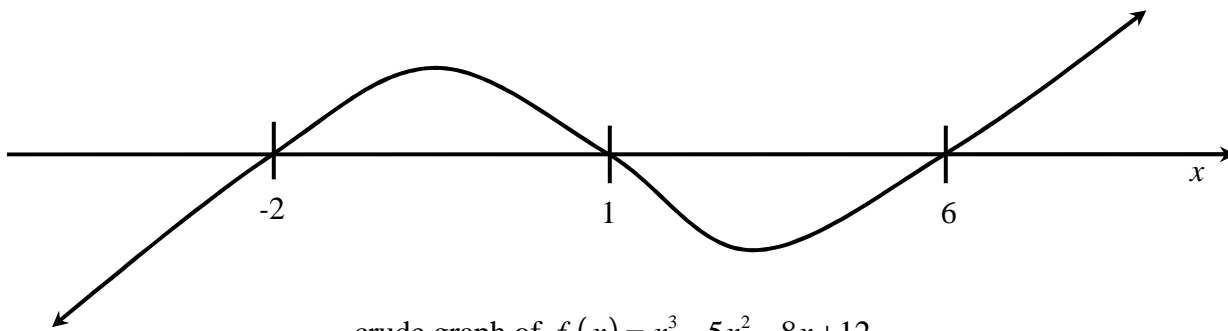
sign chart for  $[x+2](x-1)\{x-6\}$

Observe that in the above chart, the sign behavior of each factor is not influenced by the presence of the other factors. And note that the product of the three signs is indicated with a word written above the product. Let's clean up this chart a bit, and focus on what it tells us about the sign behavior of the  $y$ -values on the graph of the function  $f(x) = x^3 - 5x^2 - 8x + 12$ .



sign chart for  $f(x) = x^3 - 5x^2 - 8x + 12 = (x+2)(x-1)(x-6)$

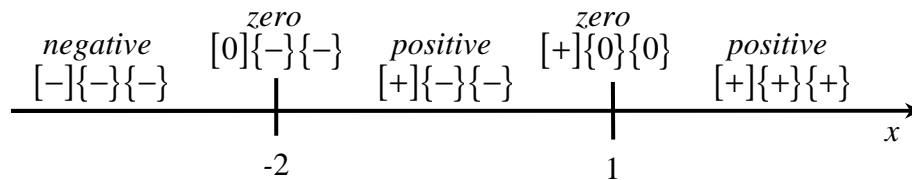
Based on this sign information, we can make a crude graph of the function  $f(x) = x^3 - 5x^2 - 8x + 12$ :



crude graph of  $f(x) = x^3 - 5x^2 - 8x + 12$

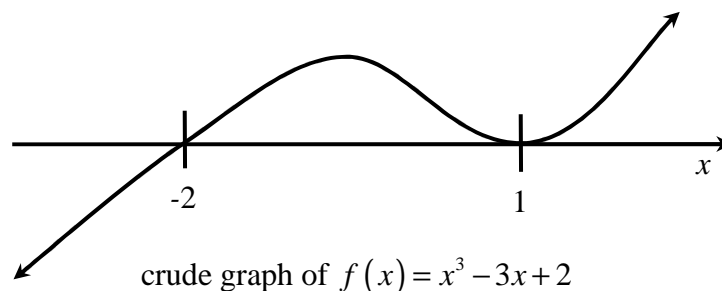
Notice that the graph of  $f$  has  $x$ -intercepts  $(-2, 0)$ ,  $(1, 0)$ , and  $(6, 0)$ , corresponding to the linear factors  $(x + 2)$ ,  $(x - 1)$ , and  $(x - 6)$  and the roots  $x = -2$ ,  $x = 1$ , and  $x = 6$ .

Example 1: Let  $f(x) = x^3 - 3x + 2 = (x + 2)\{x - 1\}\{x - 1\} = (x + 2)\{x - 1\}^2$ . Observe that  $f$  is a product of three linear factors, but two of them correspond to the same root,  $x = 1$ . The sign chart is:



sign chart for  $[x + 2]\{x - 1\}\{x - 1\}$

Based on this sign information, we can make a crude graph of the function  $f(x) = x^3 - 3x + 2$ :

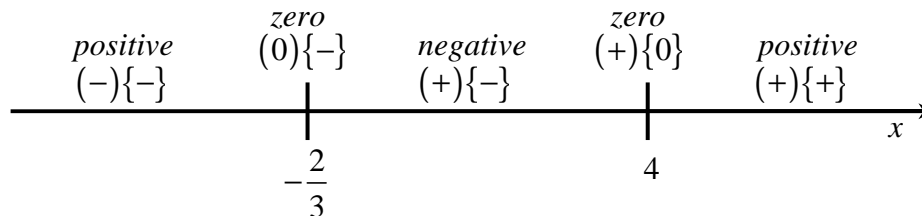


crude graph of  $f(x) = x^3 - 3x + 2$

Notice that the graph of  $f$  has  $x$ -intercepts  $(-2, 0)$  and  $(1, 0)$ , corresponding to the linear factors  $(x + 2)$  and  $(x - 1)$  and the roots  $x = -2$  and  $x = 1$ . Notice also that we can see very clearly the effect of having the linear factor  $(x - 1)$  appearing twice in the factorization for  $f$ . It causes the graph to touch but not cross the  $x$ -axis at the point  $(1, 0)$ .

Example 2: Let  $g(x) = 3x^2 - 10x - 8$ . This function can be factored as  $g(x) = (3x + 2)(x - 4)$ . But notice that the factor  $(3x + 2)$  is not in the form  $(x - r)$ ; it is not what we have been calling a “linear factor”. It can be put into that form by factoring out the number 3:  $(3x + 2) = 3\left(x + \frac{2}{3}\right)$ . Plugging this

back into the function  $g$ , we have  $g(x) = 3\left(x + \frac{2}{3}\right)(x - 4)$ . Written this way, we see that  $g$  is a product of two linear factors and a constant. For clarity, let's give each linear factor a distinct type of brackets, writing the function as  $g(x) = 3\left(x + \frac{2}{3}\right)\{x - 4\}$ . The constant 3 in front will have no effect on the sign of the function. The sign chart for the function is shown below.



$$\text{sign chart for } g(x) = 3x^2 - 10x - 8 = 3\left(x + \frac{2}{3}\right)\{x - 4\}$$

## 5. Exercises

- 1) Make a sign chart for the function  $y = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$ . This function will be useful in a future class example.
- 2) Make a sign chart for the function  $y = 2x^2 - 2x - 4 = 2(x^2 - x - 2) = 2(x + 1)(x - 2)$ . This function will be useful in homework problem 5.1#12.
- 3) Make a sign chart for the function  $y = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x + 4)(x - 2)$ . This function will be useful in homework problem 5.2#12.
- 4) Make a sign chart for the function  $y = -.0032x + 3.6 = -.0032(x - 1125)$ . This function will be useful in homework problem 5.1#36. (Hint: the constant term  $-.0032$  in front contributes a negative sign to the sign chart.)
- 5) Make a sign chart for the function  $y = 2ax + b = 2a\left(x + \frac{b}{2a}\right)$ . This function will be useful in homework problem 5.2#30. (Hint: the constant term  $2a$  in front contributes a sign to the sign chart, but because we don't know the value of  $a$ , we do not know whether that sign is positive or negative. That doesn't matter, though, because in this exercise what we're primarily interested in is the  $x$ -value where the sign of  $g$  changes. We won't be able to tell whether the sign is changing from negative to positive, or from positive to negative, but we will be able to tell where the change occurs.)
- 6) Make a sign chart for the function  $y = x^3 - 6x^2 + 9x - 4 = (x - 1)(x - 1)(x - 4)$ .