Ten Step Method for Graphing Rational Functions with Calculus

Math 163A Calculus Sections 02 and 04 (Barsamian)

Step 1: If $x=0$ is in the domain, find $f(0)$.

Step 2: Check for symmetries.

Step 3: Determine the end behavior (horizontal asymptote? slant asymptote? power function?) by deciding which of the following three cases applies.

case 1: degree of numerator < degree of denominator

In this case, the end behavior will resemble $y=\frac{1}{x^m}$. So the line $y=0$ will be a horizontal asymptote.

case 2: degree of numerator = degree of denominator

In this case, the end behavior will resemble $y=\frac{ax^m}{bx^m}=\frac{a}{b}x^0=\frac{a}{b}$. So the line $y=\frac{a}{b}$ will be a horizontal asymptote.

case 3: degree of numerator > degree of denominator

case 3a: degree of numerator = 1 + degree of denominator

In this case, the the end behavior will resemble $y=\frac{ax^{m+1}}{bx^m}=\frac{a}{b}x$. So, the line $y=\frac{a}{b}x$ will be a slant asymptote.

case 3b: degree of numerator ≥ 2 + degree of denominator

In this case, the end behavior will resemble $y=x^m$, for some integer $m\geq 2$.

Step 4: Make a sign chart for f. To do this, factor the function (both numerator and denominator). This will enable you to determine the domain of the function, and it will also tell you the important x-values for f. (If a linear factor $(x-r)$ appears in the factorization, then the number r is an important x-value.) Then, put all the important x-values on a number line. In each region and at each important x-value, determine whether f is positive, negative, zero, or undefined.

Step 5: Locate vertical asymptotes, holes, and x-intercepts by examining the linear factors in the factorization. For each linear factor $(x-r)$ in the factorization, decide which of the following five cases applies.

case 1: The linear factor $(x-r)$ appears in the numerator but not in the denominator.

In this case, the graph will have an x-intercept at $x=r$.

case 2: The linear factor $(x-r)$ appears in both the numerator and denominator but with a larger exponent in the numerator.

In this case, the graph will cross the x-axis at $x=r$, but there will be a hole at the crossing.

case 3: The linear factor $(x-r)$ appears in both the numerator and in the denominator and with equal exponents.

In this case, the graph will have a hole at $x=r$.

case 4: The linear factor $(x-r)$ appears in both the numerator and denominator but with a smaller exponent in the numerator.

In this case, the graph will have a vertical asymptote at $x=r$.

case 5: The linear factor $(x-r)$ appears in the denominator only.

In this case, the graph will have a vertical asymptote at $x=r$.

Step 6: Find the derivative f' and factor its numerator and denominator. This will tell you the important x-values for f'.

Step 7: Make a sign chart for f' to determine the x-values for which f' is positive, negative, zero, or undefined. This will tell you the x-values for which f is increasing, decreasing, or horizontal. The x-values for which $f'(x)=0$ or $f'(x)$ is undefined are called *critical numbers* of the function f. Plug the critical numbers into f to find the *critical values* and *critical points*.

Step 8: Find the second derivative f'' and factor its numerator and denominator. This will tell you the important x-values for f''.

Step 9: Make a sign chart for f'' to determine the x-values for which f'' is positive, negative, zero, or undefined. This will tell you the x-values for which f is concave up, concave down, or not concave. The x-values at which the graph of f changes concavity (that is, where f'' changes sign) are called *points of inflection*. Plug these x-values into f to find the y-values of the points of inflection.

Step 10: Based on the analysis in steps 1 through step 9, sketch the graph of f.