Difficult Derivatives for Four Exercises from Section 5.4

Math 163A sections 02 and 04 (Barsamian)

Exercise 5.4 #17, 18, 19, 20 ask you to sketch the graphs of rational functions. These are difficult problems, made even more difficult by the fact that the derivatives and second derivatives of these functions get quite messy. Here are the results of the derivative calculations, along with a bit of commentary that will help you when making the sign charts for the various functions and derivatives.

5.4#18 We start with this exercise because it is the most simple of the four.

$$f(x)=\frac{1}{x^2+1}$$

$$f'(x)=\frac{-2x}{(x^2+1)^2}=\frac{-2}{(x^2+1)^2}\cdot\frac{(x-0)}{1}$$

$$f''(x)=\frac{2(3x^2-1)}{(x^2+1)^3}=\frac{6\left(x^2-\frac{1}{3}\right)}{(x^2+1)^3}=\frac{6}{(x^2+1)^3}\cdot\frac{\left(x+\frac{1}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)}{1}$$

Discussion of the factorization of f

Observe that the graph of the function $y=x^2+1$ is a standard parabola that has been moved up one unit. The graph never touches the x-axis. Therefore, the function $y=x^2+1$ does not have any roots, and cannot be factored into linear factors. Because the y-values are always positive, when one computes $\frac{1}{x^2+1}$, the result will always be $\frac{1}{pos}=pos$. Because the function f does not contain any linear factors, the sign chart for f will not have any important x-values on it. The sign for f will always be positive, because of the $\frac{1}{x^2+1}$ term.

Discussion of the factorization of f'

The factorization of f' contains only one linear factor, $(x-0)$. Therefore, the sign chart for f' will have only one important x-value: $x=0$. The sign chart should have an empty pair of parentheses, $(\ )$, in each region, corresponding to the linear factor linear factor $(x-0)$. The $\frac{-2}{(x^2+1)^2}$ term in the factorization of f' is always negative. Its only contribution to the sign chart will be an unchanging minus sign placed in front of every empty pair of parentheses, $-(\ )$.

Discussion of the factorization of f''

The factorization of f'' contains two linear factors, $\left(x+\frac{1}{\sqrt{3}}\right)$ and $\left(x-\frac{1}{\sqrt{3}}\right)$. Therefore, the sign chart for f'' will have two important x-values: $x=-\frac{1}{\sqrt{3}}$ and $x=\frac{1}{\sqrt{3}}$. The sign chart should have two empty pairs of parentheses in each region ,$(\ )(\ )$, corresponding to the two linear factors. The $\frac{6}{\left(x^2+1\right)^3}$ term in the factorization of f'' is always positive. Its only contribution to the sign chart will be an unchanging plus sign placed in front of every $(\ )(\ )$.

5.4#17 This exercise is slightly more complicated than the previous one.

$$f(x)=\frac{x}{x^2+1}=\frac{1}{x^2+1}\cdot\frac{(x-0)}{1}$$

$$f'(x)=\frac{-\left(x^2-1\right)}{\left(x^2+1\right)^2}=\frac{-1}{\left(x^2+1\right)^2}\cdot\frac{(x+1)(x-1)}{1}$$

$$f''(x)=\frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}=\frac{2}{\left(x^2+1\right)^3}\cdot\frac{\left(x+\sqrt{3}\right)(x-0)\left(x-\sqrt{3}\right)}{1}$$

Discussion of the factorization of f

The factorization of f contains only one linear factor, $(x-0)$. Therefore, the sign chart for f will have only one important *x*-value: $x=0$. The sign chart should have an empty pair of parentheses in each region, corresponding to the linear factor linear factor $(x-0)$. The $\frac{1}{x^2+1}$ term in the factorization of f is always positive. Its only contribution to the sign chart will be an unchanging plus sign placed in front of every empty pair of parentheses.

Discussion of the factorization of f'

The factorization of f' contains two linear factors, $(x+1)$ and $(x-1)$. Therefore, the sign chart for f'' will have two important *x*-values: $x=-1$ and $x=1$. The sign chart should have two empty pairs of parentheses in each region , $(\ )(\ )$, corresponding to the two linear factors. The $\frac{-1}{\left(x^2+1\right)^2}$ term in the factorization of f' is always negative. Its only contribution to the sign chart will be an unchanging minus sign placed in front of every $(\ )(\ )$.

Discussion of the factorization of f''

The factorization of f'' contains three linear factors, $\left(x+\sqrt{3}\right)$, $(x-0)$, and $\left(x-\sqrt{3}\right)$. Therefore, the sign chart for f'' will have three important *x*-values: $x=-\sqrt{3}$, $x=0$ and $x=\sqrt{3}$. The sign chart should have three empty pairs of parentheses in each region , $(\ )(\ )(\ )$, corresponding to the three linear factors. The $\frac{2}{\left(x^2+1\right)^3}$ term in the factorization of f'' is always positive. Its only contribution to the sign chart will be an unchanging plus sign placed in front of every $(\ )(\ )(\ )$.

5.4#19alternate I suggest an alternate version of this problem, using a "1" instead of a "4" to make the function more similar to the other three functions.

$$f(x)=\frac{x}{x^2-1}=\frac{1}{(x+1)(x-1)}$$

$$f'(x)=\frac{-2x}{\left(x^2-1\right)^2}=-2\cdot\frac{(x-0)}{\left((x+1)(x-1)\right)^2}=-2\cdot\frac{(x-0)}{\left[(x+1)^2\right]\left[(x-1)^2\right]}$$

$$f''(x)=\frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}=2\left(3x^2+1\right)\cdot\frac{1}{\left((x+1)(x-1)\right)^3}=2\left(3x^2+1\right)\cdot\frac{1}{(x+1)^3(x-1)^3}$$

Discussion of the factorization of f

The factorization of f contains two linear factors, $(x+1)$ and $(x-1)$. Therefore, the sign chart for f will have two important x-values: $x=-1$ and $x=1$. The sign chart should have two empty pairs of parentheses in each region , $\frac{1}{(\ )(\ )}$, corresponding to the two linear factors. Observe that these linear factors will cause asymptotes in the graph of f !!

Discussion of the factorization of f'

The factorization of f' contains three linear factors, $(x+1)$, $(x-0)$, and $(x-1)$. Therefore, the sign chart for f' will have three important x-values: $x=-1$, $x=0$ and $x=1$. The sign chart should have three empty pairs of parentheses in each region , $\frac{(\ )}{[\ ][\ ]}$, corresponding to the three linear factors. The square brackets indicate that the linear factors inside have been raised to an even power, so these square brackets will only be filled with plus signs and zeroes, never minus signs. The -2 term in front of the factorization of f' is always negative. Its only contribution to the sign chart will be an unchanging minus sign placed in front of every $\frac{(\ )}{[\ ][\ ]}$.

Discussion of the factorization of f''

The factorization of f'' contains two linear factors, $(x+1)$ and $(x-1)$. Therefore, the sign chart for f'' will have two important x-values: $x=-1$ and $x=1$. The sign chart should have two empty pairs of parentheses in each region , $\frac{1}{(\ )(\ )}$, corresponding to the two linear factors. The $2\left(3x^2+1\right)$ term in front of the factorization of f'' is always positive. Its only contribution to the sign chart will be an unchanging plus sign placed in front of every $\frac{1}{(\ )(\ )}$.

5.4#20 This is the hardest exercise. Bummer!

$$f(x)=\frac{x}{x^2-1}=\frac{(x-0)}{(x+1)(x-1)}$$

$$f'(x)=\frac{-(x^2+1)}{(x^2-1)^2}=-(x^2+1)\cdot\frac{1}{((x+1)(x-1))^2}=-(x^2+1)\cdot\frac{1}{\left[(x+1)^2\right]\left[(x-1)^2\right]}$$

$$f''(x)=\frac{2x(x^2+3)}{(x^2-1)^3}=2(x^2+3)\cdot\frac{(x-0)}{((x+1)(x-1))^3}=2(x^2+3)\cdot\frac{(x-0)}{(x+1)^3(x-1)^3}$$

Discussion of the factorization of *f*

The factorization of *f* contains three linear factors, $(x+1)$, $(x-0)$, and $(x-1)$. Therefore, the sign chart for *f* will have three important *x*-values: $x=-1$, $x=0$ and $x=1$. The sign chart should have three empty pairs of parentheses in each region , $\frac{(\ )}{(\ )(\ )}$, corresponding to the three linear factors. Observe that two of these linear factors will cause asymptotes in the graph of *f* !!

Discussion of the factorization of f'

The factorization of f' contains two linear factors, $(x+1)$ and $(x-1)$. Therefore, the sign chart for f'' will have two important *x*-values: $x=-1$ and $x=1$. The sign chart should have two empty pairs of parentheses in each region , $\frac{1}{[\ ][\ ]}$, corresponding to the two linear factors. The square brackets indicate that the linear factors inside have been raised to an even power, so these square brackets will only be filled with plus signs and zeroes, never minuses. The $-(x^2+1)$ term in front of the factorization of f' is always negative. Its only contribution to the sign chart will be an unchanging minus sign placed in front of every $\frac{1}{[\ ][\ ]}$.

Discussion of the factorization of f''

The factorization of f'' contains three linear factors, $(x+1)$, $(x-0)$, and $(x-1)$. Therefore, the sign chart for f'' will have three important *x*-values: $x=-1$, $x=0$ and $x=1$. The sign chart should have three empty pairs of parentheses in each region , $\frac{(\ )}{(\ )(\ )}$, corresponding to the three linear factors. The $2(x^2+3)$ term in front of the factorization of f'' is always positive. Its only contribution to the sign chart will be an unchanging plus sign placed in front of every $\frac{(\ )}{(\ )(\ )}$.