Transformations of Graphs

review of background material for Math 163A (Barsamian)

Contents

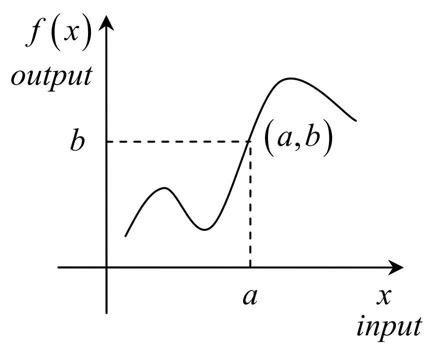
1. Introduction

This document discusses four types of common transformations of graphs that we will encounter this quarter in Math 163A: vertical addition, horizontal subtraction, vertical multiplication, and horizontal division. Such transformations are covered in courses that are prerequisites for 163A, but lower-level courses sometimes treat them only superficially. The purpose of this document is to provide a detailed review.

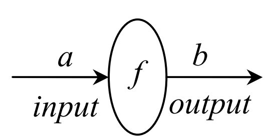
All of the examples discussed are from the list of Math 163A homework problems from Chapters 2 and 3. Furthermore, all of the Chapter 2 and 3 homework problems that involve these four types of transformations are listed in this document according to the type of transformations that they use. The discussion focusses on the "key points" of a graph, and transformations of those key points. Ironically, the document contains no graphs. There are three reasons for this: (i) the author is lazy, (ii) you will learn a lot by making your own graphs as you read, and (iii) omitting the graphs helps emphasize the importance of key points. If you take care to transform the key points correctly first, then drawing the transformed graphs easy.

2. Preliminaries: "Suppose that the graph of a function f is given."

When these words are written, or even just implied, you should visualize a picture like the one shown below.



You should imagine some point (a,b) on the graph of f. This point could be a famous point, or it could be just any arbitrary point. Either way, the fact that the point is on the graph of f tells us that whenever the number "a" is used as input to the function f, the resulting output will be the number "b". This fact could be conveyed in mathematical symbols by the equation $b = f(a)$; it can be conveyed in pictures with the machine diagram shown below.



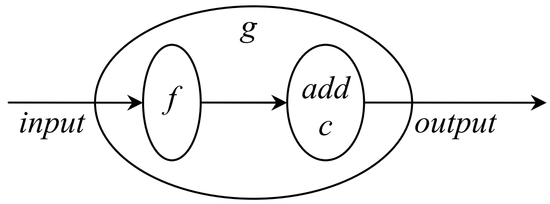
Any time we see a picture of a machine representing the function *f*, we can add to the picture the input "*a*" and the corresponding output "*b*" flanking the machine.

3. Vertical addition

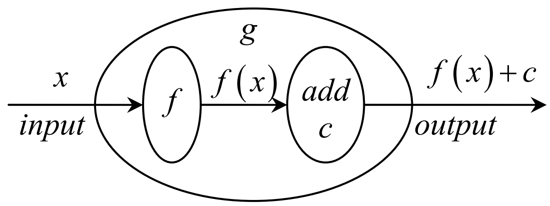
3.1. Discussion

Suppose that the graph of a function f is given, and that a new function g is defined by the equation $g(x) = f(x) + c$, where "c" is some real number. How would the graph of g look?

To start with, remember that because of the phrase "Suppose that the graph of a function f is given…", we should imagine that there is some point (a,b) on the graph of f. Next, let's make a machine diagram for the function *g*.

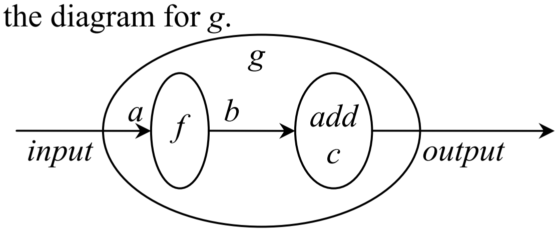


To convince yourself that this is the correct machine, consider what happens when we use an "*x*" as input:

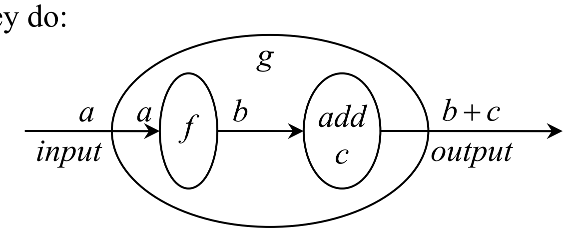


When an "*x*" is used as input to the function *g*, the resulting output is $f(x)+c$. But the output of the function *g* when an *x* is used as input is also represented by the symbol symbol $g(x)$. In other words, $g(x)=f(x)+c$. This is correct.

Now, remember that there is a hypothetical point (a,b) on the graph of *f*, and that because of that fact, anytime we see a machine diagram for the function *f*, we can put an "*a*" and "*b*" on the diagram, flanking the *f*. Let's do that to the diagram for *g*.



By looking at this diagram, we can see what must happen in the rest of the machine in order for the "*a*" and "*b*" to show up where they do:



We see that if an "*a*" is used as input to the function *g*, the resulting output is "$b+c$". That tells us that the point $(a,b+c)$ will be on the graph of *g*. This last observation is worth making very clear: if the point (a,b) is on the graph of *f*, then $(a,b+c)$ will be a point on the graph of *g*.

In conclusion, if the graph of a function f is given and a new function *g* is defined by the equation $g(x)=f(x)+c$, then the graph of *g* is obtained by adding the number *c* to the vertical coordinates of all of the points on the graph of *f*. For that reason, we will refer to a transformation of the type $g(x)=f(x)+c$ as "vertical addition".

3.2. Worked example involving vertical addition

2.2 #3 In this exercise, we have to figure out which of the six given graphs corresponds to the function $y = x^2 - 3$. We can treat this as a transformation problem. The graph of $y = x^2$ is a parabola facing up. It has five key points, $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$. The graph of $y = x^2 - 3$ is obtained by transforming the graph of $y = x^2$ by a vertical addition of -3. That means that the graph of $y = x^2 - 3$ will be a parabola facing up, with key points $(-2,1)$, $(-1,-2)$, $(0,-3)$, $(1,-2)$, and $(2,1)$. In other words, it will look like a basic parabola, but moved down 3 units. Graph "*d.*" in the book fits that description. So the answer to 2.2#3 is "*d.*"

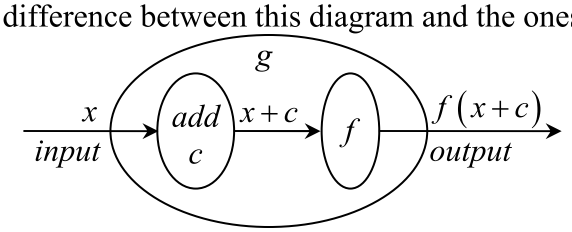
3.3. Homework exercises involving vertical addition

3.2 #21, 25

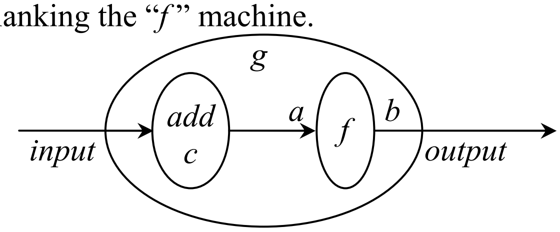
4. Horizontal subraction

4.1. Discussion

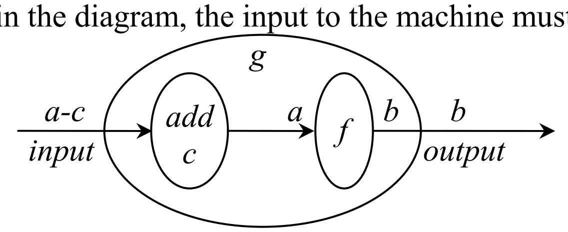
Suppose that the graph of a function f is given, and that a new function *g* is defined by the equation $g(x) = f(x+c)$, where "*c*" is some real number. How would the graph of g look? As we did in the case of "vertical addition", we start by making a machine diagram for the function *g*. A sample input "*x*" has been shown on the diagram so that we can confirm that this machine does produce the correct output. (Notice that there is a difference between this diagram and the ones from the previous section!)



Remember that there is a hypothetical point (a,b) on the graph of f and because of this, we can put an "*a*" and "*b*" on the diagram, flanking the "*f*" machine.



When we fill in the empty spots in this diagram, we observe an important phenomenon: in order for an "*a*" to show up where it does in the diagram, the input to the machine must be "*a-c*".



So, if an "*a-c*" is used as input to the function *g*, the resulting output is "*b*". That tells us that the point $(a-c,b)$ will be on the graph of *g*. This last observation is worth making very clear: if the point (a,b) is on the graph of *f*, then the $(a-c,b)$ will be a point on the graph of *g*.

In conclusion, if the graph of a function f is given, and a new function g is defined by the equation $g(x) = f(x+c)$, then the graph of g is obtained by subtracting the number c from the horizontal coordinates of all of the points on the graph of f. For that reason, we will refer to a transformation of the type $g(x) = f(x+c)$ as "horizontal subtraction".

4.2. Worked example involving horizontal subtraction

Example: Graph the function $y = (x+3)^2$ using transformations.
Solution: The graph of $y = x^2$ is a parabola facing up. It has five key points, $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$. The graph of $y = (x+3)^2$ is obtained by transforming the graph of $y = x^2$ by a horizontal subtraction of 3. That means that the graph of $y = (x+3)^2$ will be a parabola facing up, with key points $(-5,4)$, $(-4,1)$, $(-3,0)$, $(-2,1)$, and $(-1,4)$. In other words, it will look like a basic parabola, but moved to the left 3 units.

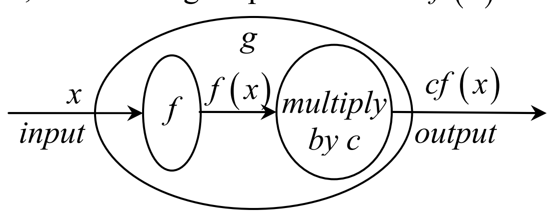
4.3. Homework exercises involving horizontal subtraction

3.2 #23

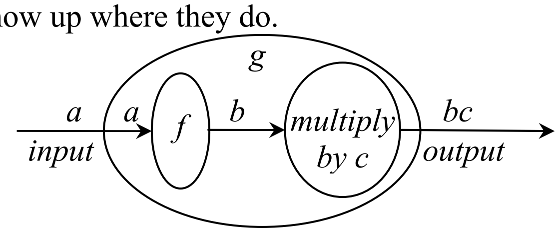
5. Vertical Multiplication

5.1. Discussion

Suppose that the graph of a function f is given, and that a new function g is defined by the equation $g(x) = cf(x)$, where "c" is some real number. How would the graph of g look? As always, we start by making a machine diagram for the function g. We confirm that this is the correct machine by confirming that when we use an "x" as input, the resulting output is in fact $cf(x)$.



As before, we put an "a" and "b" on the diagram, flanking the function f, to represent the fact that the point (a,b) is on the graph of f, and then figure out what must happen in the rest of the machine in order for the "a" and "b" to show up where they do.



If an "a" is used as input to the function g, the resulting output is "bc". That tells us that the point (a,bc) will be on the graph of g. So if the point (a,b) is on the graph of f, then the (a,bc) will be a point on the graph of g.

Thus, if the graph of a function f is given and a new function g is defined by the equation $g(x)=cf(x)$, then the graph of g is obtained by multiplying the vertical coordinates of all of the points on the graph of f by the number c. For that reason, we will refer to a transformation of the type $g(x)=cf(x)$ as "vertical multiplication".

5.2. *Worked example involving vertical multiplication*

2.2#25 Given a graph of the function $y=f(x)$, we must sketch the graph of $y=-f(x)$. The graph of f has three key points, $(-3,-2)$, $(-1,4)$, and $(5,0)$. The graph of $y=-f(x)$ is obtained by transforming the graph of $y=f(x)$ by a vertical multiplication by -1. That means that the graph of $y=-f(x)$ will have key points $(-3,2)$, $(-1,-4)$, and $(5,0)$. In other words, it will look like the original graph flipped upside down.

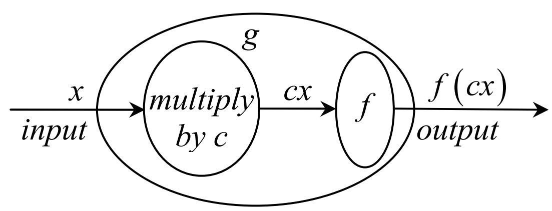
5.3. *Homework exercises involving vertical multiplication*

2.2 #1, 2, 37, 39, 41a
3.2 #25

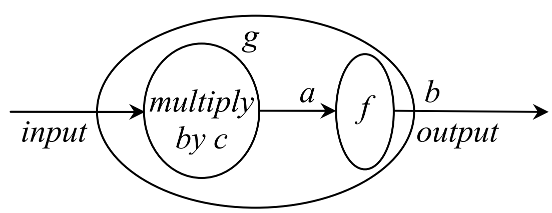
6. Horizontal Division

6.1. *Discussion*

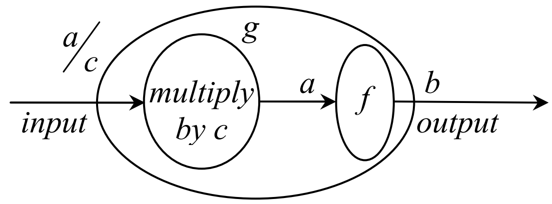
Suppose that the graph of a function f is given, and that a new function g is defined by the equation $g(x)=f(cx)$, where "c" is some non-zero real number. As always, we want to figure out how the graph of g looks, and we start by making a machine diagram for the function g, showing a sample "x" as input.



We put an "a" and "b" on the diagram, flanking the function f, to represent the fact that the point (a,b) is on the graph of f.



Next we figure out what must happen in the rest of the machine in order for the "a" and "b" to show up where they do. When we fill in the empty spots in this diagram, we observe an important phenomenon: in order for an "a" to show up where it does in the diagram, the input to the g machine must be a/c.



So, if an a/c is used as input to the function *g*, the resulting output is "*b*". That tells us that the point $\left(a/c, b\right)$ will be on the graph of *g*. Summarizing: if the point (a,b) is on the graph of *f*, then the $\left(a/c, b\right)$ will be a point on the graph of *g*. Thus, if the graph of a function f is given, and a new function *g* is defined by the equation $g(x) = f(cx)$, then the graph of *g* is obtained by dividing the horizontal coordinates of all of the points on the graph of *f* by the number *c*. For that reason, we will refer to a transformation of the type $g(x) = f(cx)$ as "horizontal division".

6.2. Worked examples involving horizontal division

Example 1: Graph the function $y = (3x)^2$ using transformations.
Solution: The graph of $y = x^2$ is a parabola facing up. It has five key points: $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$. The graph of $y = (3x)^2$ is obtained by transforming the graph of $y = x^2$ by a horizontal division of 3. That means that the graph of $y = (3x)^2$ will be a parabola facing up, with key points: $\left(-2/3, 4\right)$, $\left(-1/3, 1\right)$, $(0,0)$, $\left(1/3, 1\right)$, and $\left(2/3, 4\right)$. In other words, it will look like a basic parabola that has been compressed horizontally by a factor of 3.

Example 2: Graph the function $y = \left(\frac{x}{3}\right)^2$ using transformations.
Solution: The graph of $y = x^2$ has five key points: $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$. The graph of $y = \left(\frac{x}{3}\right)^2 = \left(\frac{1}{3} \bullet x\right)^2$ is obtained by transforming the graph of $y = x^2$ by a horizontal division by $1/3$. That means that we should divide all of the *x*-values by $1/3$. In other words, we can simply multiply all of the *x*-values by 3. So, the graph of $y = \left(\frac{x}{3}\right)^2$ will be a parabola facing up, with key points $(-6,4)$, $(-3,1)$, $(0,0)$, $(3,1)$, and $(6,4)$. It will look like a basic parabola that has been streched horizontally by a factor of 3.

6.3. Homework exercises involving horizontal division

2.2 #27, 33, 35, 41b

7. Pairs of transformations in which order does not matter

7.1. One horizontal & one vertical transformation

Suppose that the graph of a function f is given, and that a new function H is defined by the expression $H(x) = f(x+c)+d$, where "c" and "d" are real numbers. How would the graph of H look? There are clearly two transformations involved – a horizontal subtraction of c and a vertical addition of d – but which one happens first? It turns out that it doesn't matter which happens first. To see why, consider two possible scenarios: one scenario in which the horizontal subtraction of c happens first and the vertical addition of d happens second, and another scenario in which the vertical addition of d happens first and the horizontal subtraction by c happens second.

Scenario 1: Horizontal subtraction of c followed by vertical addition of d.

Suppose that the graph of a function f is given. Define a new function g by $g(x) = f(x+c)$. The graph of g is obtained from the graph of f by subtracting the number "c" from the horizontal coordinates of all the points on the graph of f. Define a new function h by $h(x) = g(x)+d$. Then the graph of h is obtained from the graph of g by adding the number "d" to the vertical coordinates of all the points on the graph of g. Now consider the hypothetical point (a,b) is on the graph of f. The point $(a-c,b)$ will be on the graph of g. This in turn means that the point $(a-c,b+d)$ will be on the graph of h. But the function h that we have presented in this scenario 1 is actually the same as the function H that was suggested in the previous paragraph. That is, $h(x) = g(x)+d = f(x+c)+d = H(x)$. We have shown that the graph of this new function H can be obtained from the graph of f by first performing a horizontal subtraction of c and then performing a vertical addition of d.

Scenario 2: Vertical addition of d followed by horizontal subtraction of c.

Suppose that the graph of a function f is given. Define a new function j by $j(x) = f(x)+d$. The graph of j is obtained from the graph of f by adding the number "d" to the vertical coordinates of all the points on the graph of f. Define a new function k by $k(x) = g(x+c)$. Then the graph of k is obtained from the graph of g by subtracting the number "c" from the horizontal coordinates of all the points on the graph of g. Now consider some point (a,b) is on the graph of f. The point $(a,b+d)$ will be on the graph of j. This in turn means that the point $(a-c,b+d)$ will be on the graph of k. But the function k that we have presented in this scenario 2 is actually the same as the function H that was introduced at the beginning of this section. That is, $k(x) = g(x+c) = f(x+c)+d = H(x)$. We have shown that the graph of this new function H can be obtained from the graph of f by first performing a vertical addition of d and then performing a horizontal subtraction of c.

In conclusion, when a function f is to be transformed by a horizontal subtraction and a vertical addition, it does not matter which transformation is done first.

In a similar way, we could show that any time a function f is to undergo any horizontal transformation and any vertical transformation, it does not matter which transformation is done first. So, we can make the following list of pairs of transformations that can be performed in either order.

Pairs of transformations that can be performed without regard to order:

- Horizontal subtraction and vertical addition
- Horizontal subtraction and vertical multiplication
- Horizontal division and vertical addition
- Horizontal division and vertical multiplication

Since it doesn't matter which of the two transformations takes place first in cases such as these, we could all just agree to always do all horizontal transformations first, and then do all vertical transformations. Agreement is not necessary, but it will help us by making our solutions to exercises always stick to a familiar form.

7.2. Worked example involving one horizontal and one vertical transformation

2.2#21 In this exercise, we must to figure out which of the six given graphs corresponds to the function $y=\sqrt{x-2}-4$. We can treat this as a transformation problem, drawing three graphs. I will discuss the properties of the three graphs, but will leave the drawing to you.

- 1st graph: $y=\sqrt{x}$ is one arm of a parabola, facing to the right. It has three key points: $(0,0)$, $(1,1)$, and $(4,2)$. The graph has 3 key points, $(0,0)$, $(1,1)$, and $(4,2)$, and has domain the set of all $x \geq 0$.
- 2nd graph: $y=\sqrt{x-2}=\sqrt{x+(-2)}$. We obtain this graph by transforming the 1st graph by horizontal subtraction of -2. That is, we should subtract -2 from all of the *x*-values. In other words, we should just add 2 to all of the *x*-values. The resulting graph is one arm of a parabola, facing to the right. It has key points $(2,0)$, $(3,1)$, and $(6,2)$ and has domain the set of all $x \geq 2$.
- 3rd graph: $y=\sqrt{x-2}-4$ is one arm of a parabola, facing to the right. It has key points $(2,-4)$, $(3,-3)$, and $(6,-2)$ and has domain the set of all $x \geq 2$.

Graph "*a*" in the book fits the description of the 3rd graph. So the answer to 2.2#21 is "*a.*"

7.3. Homework exercises involving one horizontal & one vertical transformation

2.2 #3
2.2 #4, 5, 9, 20, 26, 29, 41c

8. Pairs of transformations in which order does matter

8.1. Two horizontal transformations

Suppose that the graph of a function f is given, and that a new function H is defined by the expression $H(x)=f(cx+d)$, where "c" and "d" are real numbers with "c" nonzero. How would the graph of H look? There are clearly two transformations involved – a horizontal division by c and a horizontal subtraction of d – but which one happens first? It turns out that it does matter which happens first. To see why, consider two possible scenarios: one scenario in which the horizontal subtraction of d happens first and the horizontal division by c happens second, and another scenario in which the horizontal division by c happens first and the horizontal subtraction of d happens second.

Scenario 1: Horizontal subtraction of d followed by horizontal division by c.

Suppose the graph of a function f is given. Define a new function g by $g(x)=f(x+d)$. The graph of g is obtained from the graph of f by subtracting d from the horizontal coordinates of all the points on the graph of f. Define a new function h by $h(x)=g(cx)$. Then the graph of h is obtained from the graph of g by dividing the horizontal coordinates of all the points on the graph of g by the number c.

What happens to the point (a,b) on the graph of f when we transform that graph to make the graph of h? The point $(a-d,b)$ will be on the graph of g. This in turn means that the point $\left(\frac{a-d}{c},b\right)$ will be on the graph of h.

Scenario 2: Horizontal division by c followed by horizontal subtraction of d.

Suppose the graph of a function f is given. Define a new function j by $j(x)=f(cx)$. The graph of j is obtained from the graph of f by dividing horizontal coordinates of all the points on the graph of f by the number "c". Define a new function k by $k(x)=j(x+d)$. Then the graph of k is obtained from the graph of j by subtracting the number "d" from the horizontal coordinates of all the points on the graph of j.

Let's now ask what happens to the point (a,b) on the graph of f when we transform that graph to make the graph of k. The point $\left(\frac{a}{c},b\right)$ will be on the graph of j. This in turn means that the point $\left(\frac{a}{c}-d,b\right)$ will be on the graph of k.

Comparison of the results of scenarios 1 and 2

We see that there is disagreement between the results of scenarios 1 and 2. In scenario 1, the point (a,b) on the graph of f ends up at $\left(\frac{a-d}{c},b\right)$ on the graph of h. In scenario 2, the point (a,b) on the graph of f ends up at $\left(\frac{a}{c}-d,b\right)$ on the graph of k. So the two scenarios do not produce the same graph. But which of the two scenarios represents the correct procedure? That is, which of them makes the correct graph for the function H?

The key to answering that question lies in examining the composition of functions in the two scenarios, being careful to use parentheses.

Scenario 1: $h(x)=g(cx)=f((cx)+d)=f(cx+d)$

Scenario 2: $k(x)=j(x+d)=f(c(x+d))=f(cx+cd)$

We see that only the function produced by Scenario 1 matches the definition $H(x) = f(cx+d)$.

In conclusion, if the graph of a function f is given, and that a new function H is defined by the expression $H(x) = f(cx+d)$, where "c" and "d" are real numbers with "c" nonzero, then the graph of H can be produced by two transformations of the graph of f, where order does matter. One should transform the graph of f by first performing a horizontal subtraction of "d", and then performing a horizontal division by "c".

8.2. Worked example involving two horizontal transformations

Example: sketch the graph of $f(x) = (3x+5)^2$.

Solution: We should perform a horizontal subtraction of 5, followed by a horizontal division by 4.

- 1st graph: $y = x^2$ This graph has five key points, $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$.
- 2nd graph: $y = (x+5)^2$ This graph has key points $(-7,4)$, $(-6,1)$, $(-5,0)$, $(-4,1)$, and $(-3,4)$.
- 3rd graph: $y = (2x+5)^2$ The key points are $\left(-7/2, 4\right)$, $(-3,1)$, $\left(-5/2, 0\right)$, $(-2,1)$, and $\left(-3/2, 4\right)$.

8.3. Two vertical transformations

In a similar way, we will show that any time a function f is to undergo two vertical transformations, it does matter which transformation is done first.

Suppose that the graph of a function f is given, and that a new function H is defined by the expression $H(x) = cf(x) + d$, where "c" and "d" are real numbers. How would the graph of H look? There are two transformations involved – a vertical multiplication by c and a vertical addition of d – but it is not clear which happens first. We will again investigate by considering two scenarios: one scenario in which the vertical multiplication by c happens first and the vertical addition of d happens second, and another scenario in which the vertical addition of d happens first and the vertical multiplication by c happens second.

Scenario 1: vertical multiplication by c followed by vertical addition of d

Suppose the graph of a function f is given. Define a new function g by $g(x) = cf(x)$. The graph of g is obtained from the graph of f by multiplying the vertical coordinates of all the points on the graph of f by c. Define a new function h by $h(x) = g(x) + d$. Then the graph of h is obtained from the graph of g by adding d to the vertical coordinates of all the points on the graph of g.

What happens to the point (a,b) on the graph of f when we transform that graph to make the graph of h? The point (a,cb) will be on the graph of g. This in turn means that the point $(a,cb+d)$ will be on the graph of h.

Scenario 2: vertical addition of d followed by vertical multiplication by c.

Suppose the graph of a function f is given. Define a new function j by $j(x) = f(x) + d$. The graph of j is obtained from the graph of f by adding d to the vertical coordinates of all the points on the graph of *f* . Define a new function *k* by $k(x) = cj(x)$. Then the graph of *k* is obtained from the graph of *j* by multiplying the vertical coordinates of all the points on the graph of *j* by *c*.

Consider what happens to the point (a,b) on the graph of *f* when we transform that graph to make the graph of *k*. The point $(a,b+d)$ will be on the graph of *j*. This in turn means that the point $(a,c(b+d))$ will be on the graph of *k*.

Comparison of the results of scenarios 1 and 2

We see that there is disagreement between the results of scenarios 1 and 2. In scenario 1, the point (a,b) on the graph of *f* ends up at $(a,cb+d)$ on the graph of *h*. In scenario 2, the point (a,b) on the graph of *f* ends up at $(a,c(b+d))$ on the graph of *k*. The two scenarios do not produce the same graph. But which of the two scenarios represents the correct procedure? That is, which of them makes the correct graph for the function *H* ? As before, the key to answering that question lies in examining the composition of functions in the two scenarios, being careful to use parentheses.

Scenario 1: $h(x) = g(x) + d = (cf(x)) + d = cf(x) + d$

Scenario 2: $k(x) = cj(x) = c(f(x) + d) = cf(x) + cd$

We see that only the function produced by Scenario 1 matches the definition of $H(x) = cf(x) + d$.

In conclusion, if the graph of a function f is given, and that a new function *H* is defined by the expression $H(x) = cf(x) + d$, where "*c*" and "*d* " are real numbers, then the graph of *H* can be produced by two transformations of the graph of *f* , where order does matter. One should transform the graph of *f* by first performing a vertical multiplication by "*c*", and then performing a vertical addition by "*d* ".

8.4. Worked example involving two vertical transformations

Example: sketch the graph of $f(x) = 3x^2 + 5$.

Solution: We perform a vertical multiplication by 3, followed by a vertical addition of 5.

- 1st graph: $y = x^2$ This graph has five key points, $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$.
- 2nd graph: $y = 3x^2$ The five key points are $(-3,4)$, $(-3,1)$, $(0,0)$, $(3,1)$, and $(6,4)$.
- 3rd graph: $f(x) = 3x^2 + 5$ The five key points are $(-3,9)$, $(-3,6)$, $(0,5)$, $(3,6)$, and $(6,9)$.

9. Three transformations

9.1. One horizontal and two vertical transformations

Suppose that the graph of a function f is given, and that a new function *g* is defined by the expression $g(x) = cf(dx) + e$, where "*c*", "*d* " and "*e* " are real numbers. Let's consider how the graph of *g* might be made using transformations of graphs. There are three transformations at work: a vertical

multiplication by "*c*", a horizontal division by "*d*", and a vertical addition of "*e*". In what order should these transformations be performed?
Because horizontal and vertical transformations do not interfere with each other and can be performed without regard to order, we have agreed upon the convention that horizontal transformations would always be done before vertical. So in the present case, the first transformation that we should perform is the horizontal division by *d*. But which of the two vertical transformations should happen first? We saw earlier that a graph for the function $H(x) = cf(x) + e$ is produced by first performing a vertical multiplication by *c*, followed by a vertical addition of *e*. That means that to graph a function of the form $g(x) = cf(dx) + e$, we should perform a horizontal division by d first, followed by a vertical multiplication by *d*, and then finally do a vertical addition of e. The full sequence of graphs that we draw would be as follows:

To make a graph of $cf(dx) + e$ by transforming a graph of *f*:

- 1st graph: $f(x)$ given graph
- 2nd graph: $f(dx)$ Transform the 1st graph by a horizontal division by *d*
- 3rd graph: $cf(dx)$ Transform the 2nd graph by vertical multiplication by *c*.
- 4th graph: $cf(dx) + e$ Transform the 3rd graph by vertical addition of e.

Another form of function involving one horizontal and two vertical and one vertical transformation is the form $cf(x+d) + e$. In this form of function, the horizontal transformation is a horizontal subtraction. The graph of this type of function would be produced by the following sequence of graphs.

- 1st graph: $f(x)$ given graph
- 2nd graph: $f(x+d)$ Transform the 1st graph by a horizontal subtraction of d
- 3rd graph: $cf(x+d)$ Transform the 2nd graph by vertical multiplication by *c*.
- 4th graph: $cf(x+d) + e$ Transform the 3rd graph by vertical addition of e.

9.2. Worked example involving one horizontal and two vertical transformations

2.2#15 Sketch the graph of $f(x) = -3x^2 + 24x - 36$.

Solution: We start by putting the function into standard form, by completing the square.

$$\begin{aligned} f(x) &= -3x^2 + 24x - 36 \\ &= -3\left(x^2 - 8x\right) - 36 \\ &= -3\left(x^2 - 8x + 16 - 16\right) - 36 \\ &= -3\left(x^2 - 8x + 16\right) + 48 - 36 \\ &= -3\left(x - 4\right)^2 + 12 \\ &= -3\left(x + (-4)\right)^2 + 12 \end{aligned}$$

This involves one horizontal transformation (a horizontal subtraction of -4) and two vertical transformations (a vertical multiplication by -3 followed by a vertical addition of 12).

- 1st graph: $y = x^2$ This graph has five key points: $(-2,4)$, $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$.
- 2nd graph: $y = \left(x + (-4)\right)^2$ The five key points are $(2,4)$, $(3,1)$, $(4,0)$, $(5,1)$, and $(6,4)$.
- 3rd graph: $y = -3\left(x + (-4)\right)^2$ The five key points are $(2,-12)$, $(3,-3)$, $(4,0)$, $(5,-3)$, and $(6,-12)$.
- 4th graph: $y = -3\left(x + (-4)\right)^2 + 12$ The five key points are $(2,0)$, $(3,9)$, $(4,12)$, $(5,9)$, and $(6,0)$.

9.3. Two horizontal and one vertical transformation

Suppose that the graph of a function f is given, and that a new function *g* is defined by the expression $g(x) = cf(dx + e)$, where "*c*", "*d*" and "*e*" are real numbers. Let's consider how the graph of *g* might be made using transformations of graphs. Observe that there are three transformations at work: a vertical multiplication by "*c*", a horizontal division by "*d*", and a horizontal subtraction of "*e*". In what order should these transformations be performed?

We have seen previously that horizontal and vertical transformations do not interfere with each other: it does not matter whether horizontal or vertical is done first. As a matter of convention, we decided (okay, I decided), that we would always do horizontal transformations before vertical transformations. So, the vertical multiplication by *c* will happen last, after the two horizontal transformations. But which of the two horizontal transformations should happen first?

Well, we saw earlier that a graph for the function $f(dx + e)$ is produced by first performing a horizontal subtraction of *e*, followed by a horizontal division by *d*. That means that in the current case, we should perform a horizontal subtraction of *e* first, followed by a horizontal division by *d*, and then finally do a vertical multiplication by *c*. The full sequence of graphs that we draw should be as follows:

To make a graph of $cf(dx + e)$ by transforming a graph of *f*:

- 1st graph: $f(x)$ given graph
- 2nd graph: $f(x + e)$ Transform the 1st graph by a horizontal subtraction of *e*
- 3rd graph: $f(dx + e)$ Transform the 2nd graph by a horizontal division by d
- 4th graph: $cf(dx + e)$ Transform the 3rd graph by vertical multiplication by *c*.

Another form of function involving two horizontal and one vertical transformation is the form $g(x) = f(dx + e) + c$. In this form of function, the vertical transformation is a vertical addition. The graph of this type of function would be produced by the following sequence of graphs.

- 1st graph: $f(x)$ given graph
- 2nd graph: $f(x + e)$ Transform the 1st graph by a horizontal subtraction of *e*
- 3rd graph: $f(dx + e)$ Transform the 2nd graph by a horizontal division by d
- 4th graph: $cf(dx + e)$ Transform the 3rd graph by vertical addition of c.

9.4. Worked example involving two horizontal and one vertical transformation

Example: Make a graph of $y = \sqrt{2 - x} - 1$ by using transformations.

Solution: We start by putting it into the standard form, $y=\sqrt{-x+2}-1$. Graphing this will require two horizontal transformation (a horizontal subtraction of 2 and a horizontal division by -1) and one vertical transformation (a vertical addition of -1).

- 1st graph: $y=\sqrt{x}$. The graph has four key points, $(0,0),(1,1),(4,2),(9,3)$, and has domain $x\in[0,\infty)$. That is, the set $\{x : x\geq 0\}$.
- 2nd graph: $y=\sqrt{x+2}$. To make this graph, transform the 1st graph by a horizontal subtraction of 2. The new graph has key points $(-2,0),(-1,1),(2,2),(7,3)$ and has domain $x\in[-2,\infty)$. That is, the set $\{x : x\geq -2\}$. In other words, the 2nd graph looks like the 1st graph, but moved 2 units to the left.
- 3rd graph: $y=\sqrt{-x+2}$. To make this graph, transform the 2nd graph by a horizontal division by -1. The new graph has key points $(-7,3),(-2,2),(1,1),(2,0)$ and has domain $x\in(-\infty,2]$. That is, the set $\{x : x\leq 2\}$. In other words, the 3rd graph looks like the 2nd graph, but flipped over the *y*-axis.
- 4th graph: $y=\sqrt{-x+2}-1$. To make this graph, transform the 3rd graph by a vertical addition of -1. The new graph has key points $(-7,2),(-2,1),(1,0),(2,-1)$ and has domain $x\in(-\infty,2]$. That is, the set $\{x : x\leq 2\}$. In other words, the 4th graph looks like the 3rd graph, but moved 1 unit down.

9.5. Homework exercises involving three transformations

2.2 #6, 7, 11, 13, 15, 17, 19, 22, 23, 24, 28
2.3 #1

10. Four transformations: two horizontal and two vertical

10.1. Discussion

Suppose that the graph of a function f is given, and that a new function *g* is defined by the expression $g(x)=cf(dx+e)+h$, where "*c*", "*d*", "*e*" and "*h*" are real numbers. To make a graph of *g* by transforming the graph of *f*, we will need to make four transformations. By convention, the two horizontal transformations will be done before the two vertical transformations. The order in which we do the two horizontal transformations is dictated by considerations like the ones we have discussed above. The same goes for the order of the two vertical transformations. The full sequence of graphs that we draw should be as follows:

To make a graph of $g(x)=cf(dx+e)+h$ by transforming a graph of *f*:

- 1st graph: $f(x)$ given graph
- 2nd graph: $f(x+e)$ Transform the 1st graph by a horizontal subtraction of e
- 3rd graph: $f(dx+e)$ Transform the 2nd graph by a horizontal division by d
- 4th graph: $cf(dx+e)$ Transform the 3rd graph by vertical multiplication by c.
- 5th graph: $cf(dx+e)+h$ Transform the 4th graph by vertical addition of h.

10.2. Worked example involving four transformations

2.2#31 make a graph of $y = -\sqrt{-4-x} - 2$ by using transformations.

Solution: We start by putting the function into the standard form, $y = -\sqrt{(-x)+(-4)} - 2$. Graphing this will require two horizontal transformation (a horizontal subtraction of -4 and a horizontal division by -1) and two vertical transformations (a vertical addition multiplication by -1 followed by a vertical addition of -2).

- 1st graph: $y = \sqrt{x}$. The graph has four key points, $(0,0),(1,1),(4,2),(9,3)$, and has domain $x \in [0,\infty)$. That is, the set $\{x : x \geq 0\}$.
- 2nd graph: $y = \sqrt{x+(-4)}$. To make this graph, transform the 1st graph by a horizontal subtraction of -4. In other words, add 4 to all of the horizontal coordinates. The new graph has key points $(4,0),(5,1),(8,2),(13,3)$ and has domain $x \in [4,\infty)$. That is, the set $\{x : x \geq 4\}$. In other words, the 2nd graph looks like the 1st graph, but moved 4 units to the right.
- 3rd graph: $y = \sqrt{(-x)+(-4)}$. To make this graph, transform the 2nd graph by a horizontal division by -1. The new graph has four key points, $(-13,3),(-8,2),(-5,1),(-4,0)$, and has domain $x \in (-\infty,-4]$. That is, the set $\{x : x \leq -4\}$. So the 3rd graph looks like the 2nd graph, but flipped across the *y*-axis.
- 4th graph: $y = -\sqrt{(-x)+(-4)}$. To make this graph, transform the 3rd graph by a vertical multiplication by -1. The new graph has key points $(-13,-3),(-8,-2),(-5,-1),(-4,0)$ and has domain $x \in (-\infty,-4]$. That is, the set $\{x : x \leq -4\}$. The 4th graph looks like the 3rd graph, but flipped upside down.
- 5th graph: $y = -\sqrt{(-x)+(-4)} - 2$. To make this graph, transform the 4th graph by a vertical addition of -2. The new graph has key points $(-13,-5),(-8,-4),(-5,1),(-4,2)$ and has domain $x \in (-\infty,-4]$. That is, the set $\{x : x \leq -4\}$. The resulting graph looks like the 4th graph, but moved down 2 units.