

## Handout 07: Ten Step Method for Graphing Rational Functions With Calculus

Math 163A Calculus Section 02 (Barsamian)

**Step 1:** If  $x = 0$  is in the domain, find  $f(0)$ .

**Step 2:** Check for symmetries.

**Step 3:** Determine the end behavior (horizontal asymptote? slant asymptote? power function?) by deciding which of the following three cases applies.

**case 1:** degree of numerator < degree of denominator

In this case, the end behavior will resemble  $y = \frac{1}{x^m}$ . So the line  $y = 0$  will be a horizontal asymptote.

**case 2:** degree of numerator = degree of denominator

In this case, the end behavior will resemble  $y = \frac{ax^m}{bx^m} = \frac{a}{b}x^0 = \frac{a}{b}$ . So the line  $y = \frac{a}{b}$  will be a horizontal asymptote.

**case 3:** degree of numerator > degree of denominator

**case 3a:** degree of numerator = 1 + degree of denominator

In this case, the end behavior will resemble  $y = \frac{ax^{m+1}}{bx^m} = \frac{a}{b}x$ . So, the line  $y = \frac{a}{b}x$  will be a slant asymptote.

**case 3b:** degree of numerator  $\geq 2$  + degree of denominator

In this case, the end behavior will resemble  $y = x^m$ , for some integer  $m \geq 2$ .

**Step 4:** Make a sign chart for  $f$ . To do this, factor the function (both numerator and denominator). This will enable you to determine the domain of the function, and it will also tell you the important  $x$ -values for  $f$ . (If a linear factor  $(x - r)$  appears in the factorization, then the number  $r$  is an important  $x$ -value.) Then, put all the important  $x$ -values on a number line. In each region and at each important  $x$ -value, determine whether  $f$  is positive, negative, zero, or undefined.

**Step 5:** Locate vertical asymptotes, holes, and  $x$ -intercepts by examining the linear factors in the factorization. For each linear factor  $(x - r)$  in the factorization, decide which of the following five cases applies.

**case 1:** The linear factor  $(x - r)$  appears in the numerator but not in the denominator.

In this case, the graph will have an  $x$ -intercept at  $x = r$ .

**case 2:** The linear factor  $(x - r)$  appears in both the numerator and denominator but with a larger exponent in the numerator.

In this case, the graph will cross the  $x$ -axis at  $x = r$ , but there will be a hole at the crossing.

**case 3:** The linear factor  $(x - r)$  appears in both the numerator and in the denominator and with equal exponents.

In this case, the graph will have a hole at  $x = r$ .

**case 4:** The linear factor  $(x - r)$  appears in both the numerator and denominator but with a smaller exponent in the numerator.

In this case, the graph will have a vertical asymptote at  $x = r$ .

**case 5:** The linear factor  $(x - r)$  appears in the denominator only.

In this case, the graph will have a vertical asymptote at  $x = r$ .

**Step 6:** Find the derivative  $f'$  and factor its numerator and denominator. This will tell you the important  $x$ -values for  $f'$ .

**Step 7:** Make a sign chart for  $f'$  to determine the  $x$ -values for which  $f'$  is positive, negative, zero, or undefined. This will tell you the  $x$ -values for which  $f$  is increasing, decreasing, or horizontal. The  $x$ -values for which  $f'(x) = 0$  are called *critical numbers* of the function  $f$ . Plug the critical numbers into  $f$  to find the *critical values* and *critical points*.

**Step 8:** Find the second derivative  $f''$  and factor its numerator and denominator. This will tell you the important  $x$ -values for  $f''$ .

**Step 9:** Make a sign chart for  $f''$  to determine the  $x$ -values for which  $f''$  is positive, negative, zero, or undefined. This will tell you the  $x$ -values for which  $f$  is concave up, concave down, or not concave. The  $x$ -values at which the graph of  $f$  changes concavity (that is, where  $f''$  changes sign) are called *points of inflection*. Plug these  $x$ -values into  $f$  to find the  $y$ -values of the points of inflection.

**Step 10:** Based on the analysis in steps 1 through step 9, sketch the graph of  $f$ .