

Unit 1: Triangles
Math 330B/539 Spring 2006 (Barsamian)

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1. Introduction

In the first unit of Math 330B, we will study triangles. Our study will be organized into three parts, each taking about a week of class time. The first part introduces the concept of triangle similarity and presents several important Euclidean geometry theorems about similarity. The second part applies ideas about area to re-prove some of the theorems from part 1. Then, some of the same styles of proof are used to prove the Pythagorean Theorem for Euclidean geometry. In the third part, we will study trigonometry. In particular, we will compare different versions of the trigonometric functions. These notes are adapted by Mark Barsamian from material originally produced by Barbara Grover and Jeff Connor. (In their material, the unit was entitled “Similarity”) The adaptation of the notes is still in progress. Please e-mail Mark any comments on the content and information about any typos that you find.

1.1. Review of axioms and theorems from Math 330A

The following are theorems of absolute geometry:

- *Theorem (Alternate Interior Angle Theorem for absolute geometry) (AIAT):* In absolute geometry, given two lines cut by a transversal, if alternate interior angles are congruent, then the two lines are parallel.
- *Theorem (Corresponding Angle Theorem for absolute geometry):* In absolute geometry, given two lines cut by a transversal, if corresponding angles are congruent, then the two lines are parallel.

There is an important recurring question in geometry. We could call it THE BIG QUESTION:

- Given a line L and a point P not on L , how many lines exist that contain P and are parallel to L ?

The list of axioms for Absolute Geometry does not include an axiom about parallel lines. But the following theorem answers THE BIG QUESTION in the case of absolute geometry

- *Theorem:* In absolute geometry, if L is a line and P is a point not on L , then there exists at least one line through P that is parallel to L .

The list of axioms for Hyperbolic geometry consists of all of the axioms for absolute geometry, plus this one, which answers THE BIG QUESTION in the case of hyperbolic geometry:

- *Hyperbolic Parallel Postulate (HPP):* In Hyperbolic geometry, if L is a line and P is a point not on L , then there exists more than one line through P that is parallel to L .

The list of axioms for Euclidean geometry consists of all of the axioms for absolute geometry, plus this one, which answers THE BIG QUESTION in the case of Euclidean geometry:

- *Euclidean Parallel Postulate (EPP), Playfair’s version:* In Euclidean geometry, if L is a line and P is a point not on L , then there exists exactly one line M through P that is parallel to L .

Consequences of the EPP:

- *Theorem:* In Euclidean geometry, if two lines are parallel to a third line, then they are parallel to each other.
- *Theorem:* In Euclidean geometry, if a line intersects one of two parallel lines, then it also intersects the other.
- *Theorem (Converse of the Alternate Interior Angle Theorem for Euclidean geometry):* In Euclidean geometry, given two lines cut by a transversal, if the two lines are parallel, then alternate interior angles are congruent.

- *Theorem (Converse of the Corresponding Angle Theorem for Euclidean Geometry):* In Euclidean geometry, given two lines cut by a transversal, if the two lines are parallel, then corresponding angles are congruent.

2. Triangle Congruence and Similarity

2.1. Introduction and definitions

In this section, we will discuss triangle congruence, and triangle similarity. We begin with definitions.

Definition 1 “triangle”

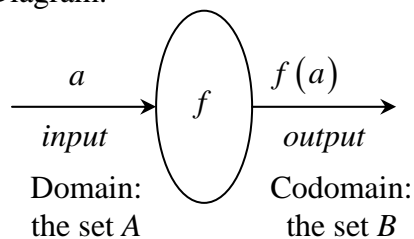
- Symbol: $\triangle ABC$
- Words: “triangle A, B, C ”
- Usage: A, B, C are non-collinear points.
- Meaning: $\overline{AB} \cup \overline{BC} \cup \overline{CA}$

Remarks:

- 1) Remember that line segments are sets of points. A union of line segments is also a set of points, and so a triangle is a set of points.
- 2) Order is not important when listing the elements of a set. So the set of points represented by the symbol $\overline{AB} \cup \overline{BC} \cup \overline{CA}$ is the same as the set of points represented by the symbol $\overline{BC} \cup \overline{CA} \cup \overline{AB}$. Therefore, the symbols $\triangle ABC$ and $\triangle BCA$ represent the same triangle. In fact, given three non-collinear points, $A, B,$ and $C,$ all six of the symbols $\triangle ABC, \triangle ACB, \triangle BAC, \triangle BCA, \triangle CAB,$ and $\triangle CBA$ represent the same triangle.

Definition 2 “function”, “domain”, “codomain”, “image”, “machine diagram”; “correspondence”

- Symbol: $f : A \rightarrow B$
- Spoken: “ f is a function that maps A to B ”
- Usage: A and B are sets. Set A is called the *domain* and set B is called the *codomain*.
- Meaning: f is a machine that takes an element of set A as input and produces an element of set B as output.
- More notation: If an element $a \in A$ is used as the input to the function $f,$ then the symbol $f(a)$ is used to denote the corresponding output. The output $f(a)$ is called *the image of a under the map f* .
- Machine Diagram:

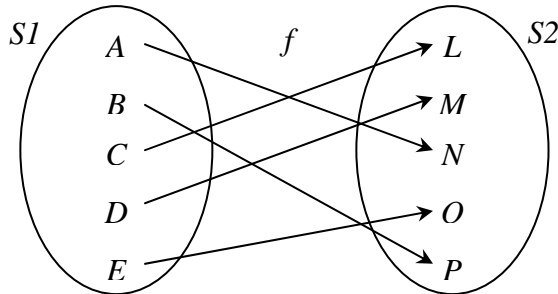


- Additional notation: If f is both *one-to-one* and *onto* (that is, if f is a *bijection*), then the symbol $f : A \leftrightarrow B$ will be used. In this case, f is called a *correspondence* between the sets A and B .

Correspondences play a key role in the concept of triangle similarity and congruence, and they will also play a key role in the concept of polygon similarity and congruence, so we should do a few examples to get more familiar with them.

Examples

- 1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the cubing function, $f(x) = x^3$. Then f is one-to-one and onto, so we could say that f is a correspondence, and we would write $f : \mathbb{R} \leftrightarrow \mathbb{R}$.
- 2) Let $S_1 = \{A, B, C, D, E\}$ and $S_2 = \{L, M, N, O, P\}$. Define a function $f : S_1 \rightarrow S_2$ by this picture:



Then we would say that f is a correspondence, and we would write $f : S_1 \leftrightarrow S_2$.

- 3) For the same example as above, we could display the correspondence more concisely:

$$\begin{aligned} A &\leftrightarrow N \\ B &\leftrightarrow P \\ C &\leftrightarrow L \\ D &\leftrightarrow M \\ E &\leftrightarrow O \end{aligned}$$

This takes up much less space, and is faster to write, than the picture. However, notice that this way of displaying the correspondence still uses a lot of space.

- 4) There is an even more concise way to display the correspondence from the above example. To understand the notation, though, we should first recall some conventions about brackets and parentheses. When displaying sets (where order is not important), curly brackets are used. For example, $S_1 = \{A, B, C, D, E\} = \{C, A, E, D, B\}$. But when displaying an ordered list, parentheses are used instead. So while the symbol $\{A, B\}$ denotes the same set as the symbol $\{B, A\}$, the symbol (A, B) denotes a different ordered pair from the symbol (B, A) . With that notation in mind, we will use the symbol below to denote the function f described in the previous examples.

$$(A, B, C, D, E) \leftrightarrow (N, P, L, M, O)$$

The parentheses indicate that the order of the elements is important, and the double arrow symbol indicates that there is a correspondence between the lists. Notice that this way of displaying the function is not as clear as the one in the previous example, but it takes up much less space.

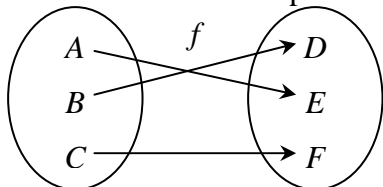
Definition 3 Correspondence between vertices of two triangles.

- Words: “ f is a correspondence between the vertices of triangles $\triangle ABC$ and $\triangle DEF$.”
- Meaning: f is a *one-to-one, onto* function with domain $\{A, B, C\}$ and codomain $\{D, E, F\}$.

Examples of correspondences between the vertices of $\triangle ABC$ and $\triangle DEF$.

- 1) $(A, B, C) \leftrightarrow (D, E, F)$
- 2) $(A, B, C) \leftrightarrow (D, F, E)$
- 3) $(B, A, C) \leftrightarrow (D, E, F)$
- 4) $(B, C, A) \leftrightarrow (D, F, E)$

Notice that the third and fourth examples are actually the same. Each could be illustrated by this figure:



If a correspondence between the vertices of two triangles has been given, then there is an automatic correspondence between any other geometric items that are defined purely in terms of those vertices. For example, suppose that we are given the following correspondence $(B, A, C) \leftrightarrow (D, E, F)$ between the vertices of $\triangle ABC$ and $\triangle DEF$. For clarity, we can display the correspondence vertically. There is a correspondence between the sides of triangle $\triangle ABC$ and the sides of $\triangle DEF$, and a correspondence between the angles of triangle $\triangle ABC$ and the angles of $\triangle DEF$, since those items are defined only in terms of the vertices.

$B \leftrightarrow D$ $A \leftrightarrow E$ $C \leftrightarrow F$	the given correspondence between vertices of $\triangle ABC$ and vertices of $\triangle DEF$
$\overline{BA} \leftrightarrow \overline{DE}$ $\overline{AC} \leftrightarrow \overline{EF}$ $\overline{CB} \leftrightarrow \overline{FD}$ $\sphericalangle BAC \leftrightarrow \sphericalangle DEF$ $\sphericalangle ACB \leftrightarrow \sphericalangle EFD$ $\sphericalangle CBA \leftrightarrow \sphericalangle FDE$	the automatic correspondence between parts of $\triangle ABC$ and parts of $\triangle DEF$

Based on the ideas of this discussion, we make the following definition.

Definition 4 “corresponding parts of two triangles”

- Words: Corresponding parts of triangles $\triangle ABC$ and $\triangle DEF$.
- Usage: A correspondence between the vertices of triangles $\triangle ABC$ and $\triangle DEF$ has been given.
- Meaning: As discussed above, there is an automatic correspondence between the sides of triangles $\triangle ABC$ and the sides of $\triangle DEF$, and also between the angle of triangles $\triangle ABC$ and the angles of $\triangle DEF$. Suppose the correspondence between vertices were $(B, A, C) \leftrightarrow (D, E, F)$. Corresponding parts would be pairs such as the pair of sides, $\overline{AC} \leftrightarrow \overline{EF}$, or the pair of angles, $\sphericalangle ACB \leftrightarrow \sphericalangle EFD$.

2.2. Congruent triangles

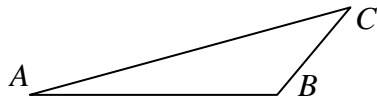
Our main focus in this chapter will be on triangle similarity. First, though, it will be useful to introduce the concept of triangle congruence. We start with the definition.

Definition 5 triangle congruence

- Symbol: $\triangle ABC \cong \triangle DEF$
- Words: $\triangle ABC$ is congruent to $\triangle DEF$.
- Meaning: There is a correspondence between vertices such that corresponding parts of the triangles are congruent.
- Additional terminology: If a correspondence between vertices of two triangles has the property that corresponding parts are congruent, then the correspondence is called a *congruence*.

Remark: Many students remember the sentence “Corresponding parts of congruent triangles are congruent” from their high school geometry course. The acronym is, of course, “CPCTC”. We see now that “CPCTC” is really a summary of the definition of triangle congruence. That is, to say that two triangles are congruent is the same as saying that corresponding parts of those two triangles are congruent. This is worth restating: CPCTC is not an axiom and it is not a theorem; it is merely shorthand for the definition of triangle congruence.

It is important to discuss notation at this point, because there is an abuse of notation common to many geometry books. In most books, the symbol $\triangle ABC \cong \triangle DEF$ is used to mean not only that triangle $\triangle ABC$ is congruent to $\triangle DEF$, but also that the correspondence between the vertices is given by $(A, B, C) \leftrightarrow (D, E, F)$. That is, in most books, the order of the vertices in the correspondence is assumed to be the same as the order of the vertices in the symbol $\triangle ABC \cong \triangle DEF$. This can cause problems. To see why, notice that given any three non-collinear points, $A, B,$ and $C,$ the symbols $\triangle ABC$ and $\triangle ACB$ represent the same triangle.



That is because a triangle is defined to be the union of three line segments, and the three line segments that make up $\triangle ABC$ are exactly the same as the three line segments that make up $\triangle ACB$. Now consider the correspondence $(A, B, C) \leftrightarrow (A, B, C)$ of vertices of triangles $\triangle ABC$ and $\triangle ACB$. Below is a list of the resulting correspondence of parts.

$A \leftrightarrow A$ $B \leftrightarrow B$ $C \leftrightarrow C$	the given correspondence between vertices of $\triangle ABC$ and vertices of $\triangle ACB$.
$\overline{AB} \leftrightarrow \overline{AB}$ $\overline{BC} \leftrightarrow \overline{BC}$ $\overline{CA} \leftrightarrow \overline{CA}$ $\sphericalangle ABC \leftrightarrow \sphericalangle ABC$ $\sphericalangle BCA \leftrightarrow \sphericalangle BCA$ $\sphericalangle CAB \leftrightarrow \sphericalangle CAB$	the automatic correspondence between parts of $\triangle ABC$ and parts of $\triangle ACB$.

Clearly, each of the pairs of corresponding parts are congruent to each other. Therefore, we would say that the correspondence $(A, B, C) \leftrightarrow (A, B, C)$ is a congruence, and triangle $\triangle ABC$ is congruent to $\triangle ACB$.

Now consider the correspondence $(A, B, C) \leftrightarrow (A, C, B)$ of vertices of triangles $\triangle ABC$ and $\triangle ACB$. Below is a list of the resulting correspondence of parts.

$A \leftrightarrow A$ $B \leftrightarrow C$ $C \leftrightarrow B$	the given correspondence between vertices of $\triangle ABC$ and vertices of $\triangle ACB$.
$\overline{AB} \leftrightarrow \overline{AC}$ $\overline{BC} \leftrightarrow \overline{CB}$ $\overline{CA} \leftrightarrow \overline{BA}$ $\sphericalangle ABC \leftrightarrow \sphericalangle ACB$ $\sphericalangle BCA \leftrightarrow \sphericalangle CBA$ $\sphericalangle CAB \leftrightarrow \sphericalangle BAC$	the automatic correspondence between parts of $\triangle ABC$ and parts of $\triangle ACB$.

Clearly, the pairs of corresponding parts are not congruent to each other. So the correspondence $(A, B, C) \leftrightarrow (A, C, B)$ is not a congruence.

So we see that triangle $\triangle ABC$ is congruent to $\triangle ACB$, but that the correspondence that should be used is $(A, B, C) \leftrightarrow (A, B, C)$, not $(A, B, C) \leftrightarrow (A, C, B)$.

In most books, the symbol $\triangle ABC \cong \triangle ACB$ is used to mean “triangle $\triangle ABC$ is congruent to $\triangle ACB$, and the correspondence used is $(A, B, C) \leftrightarrow (A, C, B)$.” This is bad, because although the triangles $\triangle ABC$ and $\triangle ACB$ are congruent, the correspondence that should be used is $(A, B, C) \leftrightarrow (A, B, C)$, not $(A, B, C) \leftrightarrow (A, C, B)$. It would be useful to have a symbol that not only indicates congruence but also tells which correspondence was used. But there is not such a symbol, and so we have to make do with the symbols available. When the particular choice of correspondence is an issue, we should write things like “ $\triangle ABC \cong \triangle ACB$ using the correspondence $(A, B, C) \leftrightarrow (A, B, C)$ ”, instead of merely writing “ $\triangle ABC \cong \triangle ACB$ ”.

Suppose that you suspected that two triangles, $\triangle ABC$ and $\triangle DEF$, were congruent. What would you have to do to verify that they actually are congruent? According to the definition of triangle congruence, you would have to first produce a correspondence between the vertices of $\triangle ABC$ and the vertices of $\triangle DEF$. Then you would have to check to see if each of the three sides of $\triangle ABC$ is congruent to the corresponding side of $\triangle DEF$, and also check to see if each of the three angles of $\triangle ABC$ is congruent to the corresponding angle of $\triangle DEF$. That is a total of six congruences that you would have to check. That is very time-consuming and in general very difficult or even impossible. Luckily, the sets of axioms used for absolute or Euclidean geometry usually include an axiom such as the following:

The SAS congruence axiom for absolute geometry: In absolute geometry, given a one-to-one correspondence between two triangles (or between a triangle and itself), if two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

This axiom basically says that you do not have to check all six congruences. It is enough to check three, as long as you check the right three.

There are a number of theorems in absolute and Euclidean geometry that are essentially about other ways of avoiding having to check all six congruences when showing that two triangles are congruent. For instance, the *SSS* congruence theorem of absolute geometry allows us to check three congruences instead of six. The *SAS* congruence theorem of absolute geometry, and the *AAS* congruence theorem for Euclidean geometry also say that checking three congruences (of the appropriate type) is sufficient. We have not yet covered any of these theorems in our course, however.

2.3. Similar triangles

Now we come to the main topic of the section: triangle similarity. We start with the definition.

Definition 6 triangle similarity

- Symbol: $\triangle ABC \sim \triangle DEF$
- Words: Triangle $\triangle ABC$ is similar to $\triangle DEF$.
- Usage: $\triangle ABC$ and $\triangle DEF$ are triangles
- Meaning: There is a correspondence between vertices such that corresponding sides have the same ratios and corresponding angles are congruent.
- Additional terminology: If a correspondence between vertices of two triangles has the property that corresponding sides have the same ratios and corresponding angles are congruent, then the correspondence is called a *similarity*.

This definition is more subtle than it looks, and so we should discuss it a bit here. Suppose that the “correspondence between vertices” is $(A, B, C) \leftrightarrow (D, E, F)$. Then the phrase “corresponding sides” refers to the following three pairs of sides: $\overline{AB} \leftrightarrow \overline{DE}$, $\overline{BC} \leftrightarrow \overline{EF}$, and $\overline{CA} \leftrightarrow \overline{FD}$. The word “ratio” refers to the ratio of the lengths of corresponding sides. There are three such ratios: $\frac{AB}{DE}$, $\frac{BC}{EF}$, and $\frac{CA}{FD}$. To say that “corresponding sides have the same ratios” means that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. So, to say that “triangle $\triangle ABC$ is similar to $\triangle DEF$ using the correspondence $(A, B, C) \leftrightarrow (D, E, F)$ ” means that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ and $\sphericalangle ABC \cong \sphericalangle DEF$, $\sphericalangle BCA \cong \sphericalangle EFD$, and $\sphericalangle CAB \cong \sphericalangle FDE$.

Confusion often arises around the types of ratios used in discussion of similar triangles. Notice that the string of equations $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ is actually equivalent to three equations $\frac{AB}{DE} = \frac{BC}{EF}$, $\frac{BC}{EF} = \frac{CA}{FD}$, and $\frac{CA}{FD} = \frac{AB}{DE}$. (Of course, only two of the three equations are needed, because of transitivity.) Each of these equations is of the form

$$\frac{\text{some side of } \triangle ABC}{\text{corresponding side of } \triangle DEF} = \frac{\text{some other side of } \triangle ABC}{\text{corresponding other side of } \triangle DEF}.$$

When we multiply both sides of the equation $\frac{AB}{DE} = \frac{BC}{EF}$ by DE and divide both sides by BC , the equation becomes $\frac{AB}{BC} = \frac{DE}{EF}$. This equation is of the form

$$\frac{\text{some side of } \triangle ABC}{\text{some other side of } \triangle ABC} = \frac{\text{corresponding side of } \triangle DEF}{\text{corresponding other side of } \triangle DEF}.$$

In other words, there are two equivalent ways of writing equations to express the fact that “corresponding sides have the same ratios”. They are

- $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, which actually means $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{BC}{EF} = \frac{CA}{FD}$, and $\frac{CA}{FD} = \frac{AB}{DE}$ (Only two of these three equations are necessary.)
- $\frac{AB}{BC} = \frac{DE}{EF}$ and $\frac{BC}{CA} = \frac{EF}{FD}$, and $\frac{CA}{AB} = \frac{FD}{DE}$ (Only two of these three equations are necessary.)

Confusion also arises around the use of the symbol for similar triangles, just as confusion arises around the use of the symbol for congruent triangles. In most books, the symbol $\triangle ABC \sim \triangle DEF$ is used to mean not only that triangle $\triangle ABC$ is similar to $\triangle DEF$, but also that the correspondence is given by $(A, B, C) \leftrightarrow (D, E, F)$. That is, in most books, the order of the vertices in the correspondence is assumed to be the same as the order of the vertices in the symbol $\triangle ABC \sim \triangle DEF$. There are problems with this usage, just as there were problems in the analogous usage of the congruence symbol, \cong . It would be useful to have a symbol that not only indicates similarity but also tells which correspondence was used. But there is not such a symbol, and so we have to make do with the symbols available. When the particular choice of correspondence is an issue, we should write things like “ $\triangle ABC \sim_{\triangle ACB}$ using the correspondence $(A, B, C) \leftrightarrow (A, B, C)$ ”, instead of merely writing “ $\triangle ABC \sim_{\triangle ACB}$ ”.

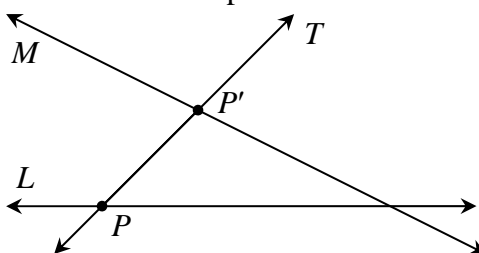
Suppose that you suspected that two triangles, $\triangle ABC$ and $\triangle DEF$, were similar. What would you have to do to verify that they actually are similar? According to the definition of triangle similarity you would have to first produce a correspondence between the vertices of $\triangle ABC$ and the vertices of $\triangle DEF$. Then you would have to check to see if each of the three angles of $\triangle ABC$ is congruent to the corresponding angle of $\triangle DEF$, and also check to see if corresponding sides have the same ratios. That is a total of three congruences and three ratios that you would have to check. That is very time-consuming and difficult.

It is natural to wonder if there might be some sort of similarity axiom—analogue to the SAS congruence axiom—that would tell us that we don’t need to check all three ratios and all three congruences in order to verify that two triangles are similar. Unfortunately, there is no such similarity axiom. But even without a similarity axiom, there are a number of similarity theorems that can be proven. These are analogous to the familiar congruence theorems. They similarity theorems tell us that we don’t have to verify all three ratios and all three congruences in order to confirm that two triangles are similar. The similarity theorems will be the focus of the next two sections.

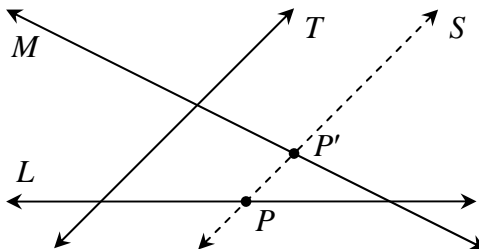
2.4. Definitions and Theorems to be discussed in class

Definition 7 Euclidean Parallel Projection

- words: f is the parallel projection of L onto M in the direction of T .
- Usage: L and M are distinct lines (meaning that they are not the same line), and T is a transversal (meaning that T intersects L and T intersects M).
- Meaning: f is a function whose domain is the set of points on L and whose codomain is the set of points on M . The operation of f is defined as follows. Let P be a point on line L that is being used as input to the function f . There are two cases to consider:
 - case 1: If P is the point of intersection of lines L and T , then the output $f(P)$ is defined to be the point P' that is the point of intersection of lines M and T .



- case 2: If P is not the point of intersection of lines L and T , then there is exactly one line S through P that is parallel to T . The output $f(P)$ is defined to be the point P' that is the point of intersection of lines M and S .



- Additional terminology: We say that *point P' is the image of point P under the parallel projection.*

Discussion of case 2:

- How do we know that there is exactly one line S through P that is parallel to T ? Answer: *EPP*.
- How do we know that line S will intersect line M ? (That is, how do we know that we will get an output?) Answer: We know that lines T and S are parallel, and that line M intersects T . A theorem of Euclidean geometry says that if a line intersects one of two parallel lines, then it also intersects the other. Therefore, M must intersect S .
- How do we know that line S is not actually the same line as line M ? (That is, how do we know that we will not get more than one output?) Answer: Again, we know that lines T and S are parallel, and that line M intersects T . Therefore, line S cannot be the same line as line M .

Theorem 1 A Euclidean parallel projection is a correspondence.

Theorem 2 (Euclidean parallel projections preserve betweenness.) In Euclidean geometry, if L is a line and $A, B,$ and C are points on L such that $A-B-C$, and $f : L \rightarrow M$ is a parallel projection, then $f(A)-f(B)-f(C)$

Theorem 3 (Euclidean parallel projections preserve congruence.) In Euclidean geometry, if L is a line and $A, B, C,$ and D are points on L such that $\overline{AB} \cong \overline{CD}$, and $f : L \rightarrow M$ is a parallel projection, then $\overline{f(A)f(B)} \cong \overline{f(C)f(D)}$.

Theorem 4 (Euclidean projections preserve basic ratios) (Basic ratio theorem) In Euclidean geometry, if L and M are lines; and $R, S,$ and T are parallel transversals that intersect line L at points $A, B,$ and C such that $A-B-C$; and $R, S,$ and T intersect line M at points A', B' and C' ; then $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.

Theorem 5 In Euclidean geometry, if two segments on a line have no point in common, then the ratio of their lengths is the same under every parallel projection.

Theorem 6 (Euclidean parallel projections preserve ratios) Euclidean geometry, for any two segments on a line, the ratio of their lengths is the same under every parallel projection.

2.5. Theorems that will be proven in the exercises

Theorem 7 In hyperbolic geometry, if two triangles are similar, then they are congruent.

Theorem 8 In spherical geometry, if two triangles are similar, then they are congruent.

Theorem 9 (AAA similarity for Euclidean geometry) In Euclidean geometry, if a correspondence between two triangles exists such that corresponding angles are congruent, then the correspondence is a similarity.

Theorem 10 (AAA congruence in hyperbolic geometry) In hyperbolic geometry, if a correspondence between two triangles exists such that corresponding angles are congruent, then the correspondence is a congruence.

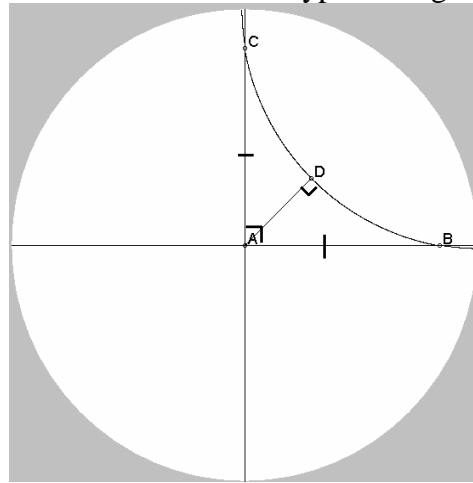
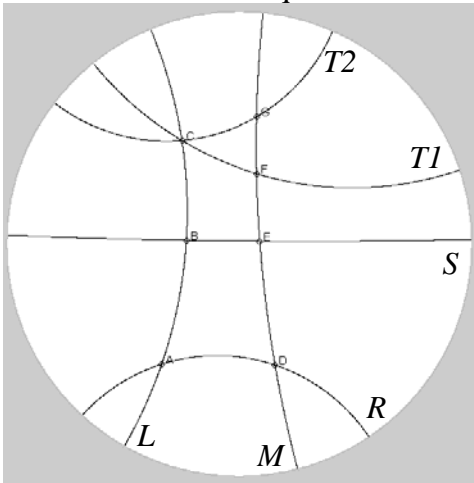
Theorem 11 (AA Similarity in Euclidean Geometry) In Euclidean geometry, if a correspondence between two triangles exists such that two pairs of corresponding angles are congruent, then the correspondence is a similarity.

Theorem 12 (SSS Similarity in Euclidean geometry) In Euclidean geometry, if a correspondence between two triangles exists such that corresponding sides are proportional, then the correspondence is a similarity.

2.6. Exercises

- 1) Use the picture at left below as an illustration for a counterexample that demonstrates that the Basic ratio theorem (Theorem 4) does not hold in hyperbolic geometry. Hint: Do an indirect proof using the method of contradiction. Assume that the Basic ratio theorem is true. Apply the basic ratio theorem to lines $L, M, R, S,$ and T_1 and points $A, B, C, D, E,$ and F . The result should be an equation involving the ratios of the lengths $AB, BC, DE,$ and EF . Call this equation #1. Then apply the basic ratio theorem again, this time to lines $L, M, R, S,$ and T_2 and points $A, B, C, D, E,$ and G . The result should be an equation involving the ratios of the lengths $AB, BC, DE,$ and EG . Call this equation #2. Use equation #1, equation #2, and transitivity to obtain a new equation involving the ratios of the lengths $DE, EF,$ and EG . Simplify this equation to obtain a new equation involving the lengths $EF,$

and EG . Show that this equation contradicts one of the axioms for hyperbolic geometry.



- 2) Let statement S be the following sentence: “If a correspondence between two triangles exists such that two pairs of corresponding angles are congruent, then in fact all three pairs of corresponding angles are congruent.” Use the picture at right above as an illustration for a counterexample that demonstrates that statement S is false in hyperbolic geometry. Hint: Do an indirect proof using the method of contradiction. Assume that statement S is true. Apply statement S to the correspondence $(A, B, C) \leftrightarrow (D, B, A)$ between vertices of triangles $\triangle ABC$ and $\triangle DBA$. The result should be some information about congruent angles. Using the area axioms, produce a statement about areas. Show that this statement contradicts one of the axioms of hyperbolic geometry.
- 3) Give a counterexample to show that the AA similarity theorem (*Theorem 11*) does not hold in hyperbolic geometry. Hint: recycle some of the ideas from the previous exercise.
- 4) Give a counterexample to show that the AA similarity theorem does not hold in spherical geometry.
- 5) Give a counterexample to show that the SSS similarity theorem (*Theorem 12*) does not hold in hyperbolic geometry.
- 6) Give a counterexample to show that the SSS similarity theorem does not hold in spherical geometry.
- 7) Prove *Theorem 7*: In hyperbolic geometry, if two triangles are similar, then they are congruent. Your proof may use any prior definition or theorem.
- 8) Prove *Theorem 8*: In spherical geometry, if two triangles are similar, then they are congruent. Your proof may use any prior definition or theorem.
- 9) Prove *Theorem 9*: (AAA similarity for Euclidean geometry). Your proof may use any prior definition or theorem.
- 10) Prove *Theorem 10*: (AAA congruence in hyperbolic geometry). Your proof may use any prior definition or theorem.
- 11) Prove *Theorem 11*: (AA Similarity in Euclidean Geometry). Your proof may use any prior definition or theorem.
- 12) Prove *Theorem 12*: (SSS Similarity in Euclidean geometry). Your proof may use any prior definition or theorem.

2.7. Proofs of Theorems

Proof of *Theorem 1* A Euclidean parallel projection is a correspondence:

Proof

Suppose that f is a parallel projection. We must show that f is a correspondence. By the definition of correspondence (*Definition 2*), that means that we have to show that f is one-to-one and onto.

Step 1: Introduce objects.

Because f is a parallel projection, we know that there are distinct lines L and M and a transversal T such that f is the parallel projection of L onto M in the direction of T . That means that $f : L \rightarrow M$.

Step 2: Prove that f is one-to-one.

We recall the definition of one-to-one.

Definition

- words: the function $f : L \rightarrow M$ is one-to-one
- meaning in symbols: $\forall P_1, P_2 \in L, \text{ IF } f(P_1) = f(P_2) \text{ THEN } P_1 = P_2.$
- meaning in words: If the outputs are the equal then the inputs are equal.
- contrapositive version in symbols: $\forall P_1, P_2 \in L, \text{ IF } P_1 \neq P_2 \text{ THEN } f(P_1) \neq f(P_2).$
- meaning in words: If the inputs are not equal then the outputs are not equal.

These two versions mean the same thing. We must prove that they are true, and we have our choice of which one we prove. I will prove the version that says, “if the outputs are the equal then the inputs are equal.”

Suppose that two input points $P_1, P_2 \in L$ have the property that the two output points are equal $f(P_1) = f(P_2)$. We must prove that $P_1 = P_2$.

Let's call the common output point Q . So $Q = f(P_1) = f(P_2)$. There are two cases to consider:

Case 1: the output point $Q = f(P_1) = f(P_2)$ lies on the transversal T .

From the definition of parallel projection, we know that the only time the output point $Q = f(P_1) = f(P_2)$ lies on the transversal T is when the input point is the point of intersection of lines L and T . So both points P_1 and P_2 must lie at the intersection of lines L and T . Therefore, $P_1 = P_2$.

Case 2: the output point $Q = f(P_1) = f(P_2)$ does not lie on the transversal T .

From the definition of parallel projection we know that because $Q = f(P_1) = f(P_2)$ does not lie on T , neither point P_1 nor P_2 lies on T . There is a line S_1 through P_1 that is parallel to transversal T , and the output point $Q = f(P_1) = f(P_2)$ lies on line S_1 . And there is a line S_2 through P_2 that is parallel to transversal T , and the output point $Q = f(P_1) = f(P_2)$ lies on line S_2 . Because we are in Euclidean geometry, we know that there can be only one line that contains Q and that is parallel to T . Therefore, the two lines S_1 and S_2 must actually be the same line, which we can call line S . So points P_1 and P_2 both lie on line S . But points P_1 and P_2 both also lie on line L . Lines L and S are not the same line, and so they can only intersect once. Therefore, points P_1 and P_2 must be the same point. That is, $P_1 = P_2$.

Therefore, $P_1 = P_2$ by the method of proof by division into cases.

We have proved that if $f(P_1) = f(P_2)$, then $P_1 = P_2$. That is, f is one-to-one.

Step 3: Prove that f is onto.

We recall the definition of onto.

Definition

- words: the function $f : L \rightarrow M$ is onto
- meaning in symbols: $\forall Q \in M, \exists P \in L$ such that $f(P) = Q$.
- meaning in words: For any element Q in the set M there is an element P in the set L such that when P is used as input to the function f , the resulting output is Q .

Let Q be any point on line M . We must prove that there is a point P on line L such that when point P is used as input to the parallel projection function f , the point Q is the resulting output.

There are two cases to consider.

Case 1: the point Q lies on the transversal T .

Let P be the point of intersection of lines T and L . By the definition of parallel projection, we know that $f(P) = Q$. We have shown that there is a point P on line L such that when point P is used as input to the parallel projection function f , the point Q is the resulting output.

Case 2: the Q does not lie on the transversal T .

We know that there is exactly one line S through Q that is parallel to T . Let P be the point of intersection of lines S and L . By the definition of parallel projection, we know that $f(P) = Q$. We have shown that there is a point P on line L such that when point P is used as input to the parallel projection function f , the point Q is the resulting output.

Therefore, by the method of proof by division into cases, we have shown that there is a point P on line L such that when point P is used as input to the parallel projection function f , the point Q is the resulting output.

End of proof

Proof of *Theorem 2* (Euclidean parallel projections preserve betweenness.) In Euclidean geometry, if L is a line and A, B , and C are points on L such that $A-B-C$, and f is a parallel projection, then $f(A)-f(B)-f(C)$.

Proof:

- 1) Suppose that L is a line and A, B , and C are points on L such that $A-B-C$, and $f : L \rightarrow M$ is a parallel projection.
- 2) Because $A-B-C$, we know that points A, B , and C are distinct points on line L . (That is, no two of them are the same point.)
- 3) Because f is a bijection (we know this from Theorem 1), we know that $f(A), f(B)$, and $f(C)$ are distinct points on line M .
- 4) Let S_A be the line containing points A and $f(A)$, let S_B be the line containing points B and $f(B)$, and let S_C be the line containing points C and $f(C)$.
- 5) From the definition of parallel projection, we know that there is some line T , transversal to lines L and M , such that the three lines S_A, S_B , and S_C are each parallel to (or the same line as) T .

- 6) Because this is Euclidean geometry, we know that the three lines S_A , S_B , and S_C are therefore parallel to each other. Or possibly two of them may be the same line. But because the three lines S_A , S_B , and S_C intersect line L at three distinct points A , B , and C , we know that no two of the three lines can be the same line. Therefore, the three lines S_A , S_B , and S_C are definitely parallel to each other.
- 7) Consider the line segment with endpoints A and $f(A)$. The symbol for this segment is $\overline{Af(A)}$. Because this segment is a subset of line S_A , and line S_A is parallel to line S_B , we know that segment $\overline{Af(A)}$ does not intersect line S_B .
- 8) Therefore, point $f(A)$ is on the same side of line S_B as point A . (by the definition of “same side”)
- 9) In a similar way, we can show that point $f(C)$ is on the same side of line S_B as point C .
- 10) Because $A-B-C$, we know that points A and C are on the opposite sides of line S_B . (by the definition of “opposite side”)
- 11) Therefore, points $f(A)$ and $f(C)$ are also on the opposite sides of line S_B . (by the betweenness axioms)
- 12) Because point $f(B)$ is the point of intersection of line S_B and line M , and the three points $f(A)$, $f(B)$, and $f(C)$ all lie on line M , the betweenness axioms tell us that therefore $f(A)-f(B)-f(C)$.

End of proof

Proof of *Theorem 3* (Euclidean parallel projections preserve congruence.) In Euclidean geometry, if L is a line and A, B, C , and D are points on L such that $\overline{AB} \cong \overline{CD}$, and $f : L \rightarrow M$ is a parallel projection, then $\overline{f(A)f(B)} \cong \overline{f(C)f(D)}$.

Proof

- 1) Suppose that L is a line and A, B, C , and D are points on L such that $\overline{AB} \cong \overline{CD}$, and $f : L \rightarrow M$ is a parallel projection.

There are two cases to consider

Case 1: Lines L and M are parallel.

- 2) In this case, the four points $A, B, f(B)$ and $f(A)$ are vertices of a parallelogram. Line segments \overline{AB} and $\overline{f(A)f(B)}$ are opposite sides of this parallelogram. Therefore, $\overline{AB} \cong \overline{f(A)f(B)}$.
- 3) Similarly, the four points $C, D, f(C)$ and $f(D)$ are vertices of a parallelogram. Line segments \overline{CD} and $\overline{f(C)f(D)}$ are opposite sides of this parallelogram. Therefore, $\overline{CD} \cong \overline{f(C)f(D)}$. By transitivity, we know that $\overline{f(A)f(B)} \cong \overline{f(C)f(D)}$.

End of case 1

Case 2: Lines L and M are not parallel.

- 4) Let S_A be the line containing points A and $f(A)$, let S_B be the line containing points B and $f(B)$, and let S_C be the line containing points C and $f(C)$, and let S_D be the line containing points D and $f(D)$.

- 5) From the definition of parallel projection, we know that there is some line T , transversal to lines L and M , such that the two lines S_A and S_B are each parallel to (or the same line as) T .
 - 6) Because this is Euclidean geometry, we know that the three lines S_A and S_B are therefore parallel to each other. Or possibly they may be the same line. But points A and B determine a line segment \overline{AB} , so it must be that A and B are distinct points. And because the lines S_A and S_B intersect line L at distinct points A and B , we know that the two lines S_A and S_B must not be the same line. Therefore, the two lines S_A and S_B are definitely parallel to each other.
 - 7) Let line V be the line through A that is parallel to M .
 - 8) Because line S_B intersects line M , and lines V and M are parallel, we know that line S_B must also intersect line V . Let E be the point of intersection of lines S_B and V .
 - 9) By a step similar to (5), we can prove that lines S_C and S_D are parallel to each other.
 - 10) Let line W be the line through C that is parallel to M .
 - 11) By a step similar to (8), we know that line S_D must also intersect line W . Let F be the point of intersection of lines S_D and W .
 - 12) Lines V and W are both parallel to M , so they are parallel to each other (because this is Euclidean geometry).
 - 13) $\sphericalangle BAE \cong \sphericalangle DCF$. (Consider line L as a transversal to parallel lines V and W . Then use the *converse of the corresponding angle theorem for Euclidean geometry*).
 - 14) $\sphericalangle ABE \cong \sphericalangle CDF$. (Consider line L as a transversal to parallel lines S_B and S_D . Then use the *converse of the corresponding angle theorem for Euclidean geometry*).
 - 15) $\triangle ABE \cong \triangle CDF$. (by steps (1), (13), (14), and the *side-angle-side axiom*)
 - 16) $\overline{AE} \cong \overline{CF}$ (by step (15) and *CPCTC*)
 - 17) $\overline{AE} \cong f(A)f(B)$ (because these segments are the opposite sides of the parallelogram determined by points $A, E, f(B)$, and $f(A)$).
 - 18) $\overline{CF} \cong f(C)f(D)$ (because these segments are the opposite sides of the parallelogram determined by points $C, F, f(D)$, and $f(C)$).
 - 19) $f(A)f(B) \cong f(C)f(D)$ (by steps (17), (18), and transitivity)
- End of case 2
- 20) $f(A)f(B) \cong f(C)f(D)$ (by steps (3), (19), and the method of proof by division into cases)
- End of proof

Proof of *Theorem 4* (Euclidean projections preserve basic ratios) (Basic ratio theorem) In Euclidean geometry, if L and M are lines; and R, S , and T are parallel transversals that intersect line L at points A, B , and C such that $A-B-C$; and R, S , and T intersect line M at points A', B' and C' ; then $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.

Proof

- 1) Suppose that L and M are lines; and R, S , and T are parallel transversals that intersect line L at points A, B , and C such that $A-B-C$; and R, S , and T intersect line M at points A', B' and C' .

- 2) Consider lines L and M and transversal line T . They fit the requirements mentioned in the definition of the Euclidean parallel projection. So there is a function $f : L \rightarrow M$ that is the parallel projection of L onto M in the direction T .
- 3) Now consider using the points A , B , and C as input to the parallel projection function f . Because of the configuration of the three parallel lines R , S , and T , we see that the corresponding output points $f(A)$, $f(B)$, and $f(C)$ would just be the three points A' , B' and C' .
- 4) Consider the ratio $\frac{AB}{BC}$ of lengths AB and BC . There are three cases to consider:

Case 1: $\frac{AB}{BC} = 1$

- 5) Then $AB = BC$.
- 6) Therefore, $\overline{AB} \cong \overline{BC}$. (by definition of congruence for line segments)
- 7) Therefore, $\overline{f(A)f(B)} \cong \overline{f(B)f(C)}$. (by *Theorem 3*)
- 8) Therefore, $\overline{A'B'} \cong \overline{B'C'}$. (by statements (3) and (7))
- 9) Therefore, $A'B' = B'C'$. (by statement (8) and the definition of congruence for line segments)
- 10) So $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. (divided the left and right sides of equation $AB = BC$ by the left and right side of equation $A'B' = B'C'$)

End of case 1

Case 2: $\frac{AB}{BC} = r$, where r is some positive rational number.

- 11) There exist positive integers p and q such that $r = \frac{p}{q}$. (because r is rational and positive)
- 12) So $\frac{AB}{BC} = \frac{p}{q}$ (by step (11) and transitivity)
- 13) $\frac{AB}{p} = \frac{BC}{q}$ (cross multiplied)
- 14) Let $x = \frac{AB}{p} = \frac{BC}{q}$. So x is some positive real number.
- 15) Notice that $AB = px$. This tells us that line segment \overline{AB} can be subdivided into p congruent segments, each of length x . Similarly, notice that $BC = qx$. This tells us that line segment \overline{BC} can be subdivided into q congruent segments, each of length x .
Consider what would happen when the endpoints of all those little congruent line segments are used as input to the parallel projection function $f : L \rightarrow M$. The input points are a batch of $p+1$ points equally spaced from A to B and a batch of $q+1$ points equally spaced from B to C , all on line L . The output points will be a bunch of points on line M . By *Theorem 2* and *Theorem 3*, the output points will be a batch of $p+1$ points equally spaced from A' to B' and a batch of $q+1$ points equally spaced from B' to C' .
- 16) Let y be the spacing of the output points on line M .

17) Then we can say that line segment $\overline{A'B'}$ can be subdivided into p congruent segments, each of length y . So $A'B' = py$. Similarly, line segment $\overline{B'C'}$ can be subdivided into q congruent segments, each of length y . So $B'C' = qy$.

18) Therefore, $\frac{A'B'}{B'C'} = \frac{py}{qy} = \frac{p}{q}$ (by step (17) and algebra).

19) Therefore, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$ (by steps (12) and (18) and transitivity).

End of case 2

Case 3: $\frac{AB}{BC} = s$, where s is some positive irrational number.

20) There exists a point d on line L between A and B such that $\frac{DB}{BC} = r$, where r is rational.

21) There exists a point e on line L such that $E-A-B$ and such that $\frac{EB}{BC} = t$, where t is rational.

22) By axioms for line segment addition, we know that $DB < AB < EB$.

23) Therefore, $\frac{DB}{BC} < \frac{AB}{BC} < \frac{EB}{BC}$. That is, $r < s < t$.

24) Consider using the five points A, B, C, D , and E on line L as input to the parallel projection function $f: L \rightarrow M$. The resulting outputs will be five points A', B', C', D' , and E' on line M .

25) Because $\frac{DB}{BC} = r$, where r is rational, the results of case 2 can be applied to say that $\frac{D'B'}{B'C'} = r$. And

similarly, because $\frac{EB}{BC} = t$, where t is rational, we can say that $\frac{E'B'}{B'C'} = t$.

26) Observe that the five input points on line L have the ordering $E-A-D-B-C$. By 0, we know that the five output points on line M will have the ordering $E'-A'-D'-B'-C'$.

27) By axioms for line segment addition, we know that $D'B' < A'B' < E'B'$.

28) Therefore, $\frac{D'B'}{B'C'} < \frac{A'B'}{B'C'} < \frac{E'B'}{B'C'}$. That is, $r < \frac{A'B'}{B'C'} < t$.

29) We never said how close points D and E are to point A . If we chose to make them very close together, then the numbers r and t will be very close to s . The inequality $r < s < t$ will always be

true, and so the inequality $r < \frac{A'B'}{B'C'} < t$ will always be true. The only way for the inequality

$r < \frac{A'B'}{B'C'} < t$ to always be true, regardless of how close the numbers r and t are to the number s , is

the value of $\frac{A'B'}{B'C'}$ to be s . Therefore, $\frac{A'B'}{B'C'} = s$.

30) Therefore, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. (by step (29) and the definition of the symbol s)

End of case 3

31) Therefore, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. (by steps (10), (19), (30), and rule of proof by division into cases)

End of proof

Proof of *Theorem 5* In Euclidean geometry, if two segments on a line have no point in common, then the ratio of their lengths is the same under every parallel projection.

Proof

- 1) Suppose that two non-intersecting segments on some line L are given, and suppose that M is some line that is not the same line as L , and that T is a line transversal to L and M .
- 2) Label the four endpoints of the two given segments as A, B, C , and D , such that $A-B-C-D$. Let $f : L \rightarrow M$ denote the parallel projection of L onto M in the direction of T . Let the symbols A', B', C' , and D' denote the outputs that result when the four points A, B, C , and D are used as input to the parallel projection function f . That is, $A' = f(A)$, etc. With the notation we have introduced, we see that our goal is to prove that $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.
- 3) Observe that $A-B-C$. Therefore, the *basic ratio theorem*, *Theorem 4*, tells us that $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.
- 4) Similarly, observe that $B-C-D$. Therefore, the *basic ratio theorem* tells us that $\frac{BC}{CD} = \frac{B'C'}{C'D'}$.
- 5) The number obtained by multiplying the left sides of these two equations equals the number obtained by multiplying the right sides of the equations. That is, $\frac{AB}{BC} \cdot \frac{BC}{CD} = \frac{A'B'}{B'C'} \cdot \frac{B'C'}{C'D'}$. Cancelling terms, we obtain $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.

End of Proof

Proof of *Theorem 6* (Euclidean parallel projections preserve ratios) In Euclidean geometry, for any two segments on a line, the ratio of their lengths is the same under every parallel projection.

Proof

- 1) Suppose that two segments on some line L are given, and suppose that M is some line that is not the same line as L , and that T is a line transversal to L and M .
- 2) Label the four endpoints of the two given segments as A, B, C , and D . Let $f : L \rightarrow M$ denote the parallel projection of L onto M in the direction of T . Let the symbols A', B', C' , and D' denote the outputs that result when the four points A, B, C , and D are used as input to the parallel projection function f . That is, $A' = f(A)$, etc. With this notation, our goal is to prove that $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.
- 3) Let E and F be points on line L such that \overline{EF} does not intersect \overline{AB} or \overline{CD} . (We could prove that such points E and F exist by careful use of the betweenness axioms.)
- 4) Because segments \overline{AB} and \overline{EF} do not intersect, *Theorem 5* tells us that $\frac{AB}{EF} = \frac{A'B'}{E'F'}$.
- 5) Similarly, because segments \overline{EF} and \overline{CD} do not intersect, we know that $\frac{EF}{CD} = \frac{E'F'}{C'D'}$.
- 6) The number obtained by multiplying the left sides of these two equations equals the number obtained by multiplying the right sides of the equations. That is, $\frac{AB}{EF} \cdot \frac{EF}{CD} = \frac{A'B'}{E'F'} \cdot \frac{E'F'}{C'D'}$. Cancelling terms, we obtain $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.

End of Proof

3. Area and Similarity; The Pythagorean Theorem

In the previous chapter, definitions were introduced for triangle congruence, triangle similarity, and Euclidean parallel projection. The theorems discussed in class explored the implications of the definition for parallel projection. The theorems proven in the exercises explored the implications of the definitions of triangle congruence and similarity. In the present chapter, we will again introduce a new definition—for the altitude of a triangle—and then study its implications.

3.1. Definitions and theorems to be discussed in class

In this section, we will establish a definition for the altitude of a triangle in absolute geometry. With the concept of altitude as a tool, we will do two things: prove the Pythagorean theorem for Euclidean geometry, and come up with a formula for the area of a triangle in Euclidean geometry.

Recall the following theorem from absolute geometry

Theorem (existence of a unique perpendicular from a point to a line in absolute geometry): In absolute geometry, if L is a line and P is a point not on L then there is exactly one point Q on line L such that $\overline{PQ} \perp L$.

That theorem allows us to make the following definition.

Definition 8 Altitude of a triangle

- words: \overline{AD} is an altitude of triangle $\triangle ABC$
- meaning: $D \in \overline{BC}$ and $\overline{AD} \perp \overline{BC}$

Because of the preceding theorem, we know that each of the three vertices of a triangle has a corresponding altitude, and that there is only one altitude for each vertex.

Remark: Keep in mind that there is no theorem like the one above in spherical geometry. In fact, there are lots of concepts of absolute geometry that have no analogues in spherical geometry. As a result, we cannot expect that a definition of an altitude for a triangle would be so straightforward in spherical geometry. In the exercises, you will be asked to discuss what goes wrong.

The definition of an altitude of a triangle shows up in the following Euclidean geometry theorem.

Theorem 13 In Euclidean geometry, in a right triangle, the altitude drawn from the right angle creates two new triangles, each of which is similar to the original triangle.

Proof

- 1) In Euclidean geometry, let $\triangle ABC$ be a triangle such that $\sphericalangle C$ is a right angle and \overline{CD} is an altitude. (We must show that $\triangle ABC \sim \triangle ACD$ and $\triangle ABC \sim \triangle CBD$.)
- 2) $\sphericalangle A \cong \sphericalangle A$ (reflexivity property of congruence)
- 3) $\sphericalangle ACB \cong \sphericalangle CDA$ (all right angles are congruent)
- 4) $\triangle ABC \sim \triangle ACD$ (by steps 1, 2, and Theorem 11, AA similarity in Euclidean geometry)
- 5) $\sphericalangle B \cong \sphericalangle B$ (reflexivity property of congruence)
- 6) $\sphericalangle ACB \cong \sphericalangle CDB$ (all right angles are congruent)
- 7) $\triangle ABC \sim \triangle CBD$ (by steps 4, 5, and Theorem 11)

End of proof

In the exercises, you will be asked to produce counterexamples to show that there could not be a theorem like Theorem 13 in hyperbolic or spherical geometry.

We will use Theorem 13 in our first proof of the Pythagorean theorem:

Theorem 14 (Pythagorean Theorem of Euclidean geometry) In Euclidean geometry, the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

Proof

- 1) In Euclidean geometry, let $\triangle ABC$ be a triangle such that $\sphericalangle C$ is a right angle. (We must show that $(AC)^2 + (BC)^2 = (AB)^2$.)
- 2) Let \overline{CD} be the altitude from vertex C .
- 3) $\triangle ABC \sim \triangle ACD$ (by steps 1, 2, and Theorem 13)
- 4) $\frac{AC}{AB} = \frac{AD}{AC}$ (by step 3, the definition of similarity, and some rearranging of the ratio)
- 5) $AD = \frac{(AC)^2}{AB}$ (by step 4 and algebra)
- 6) $\triangle ABC \sim \triangle CBD$ (by steps 1, 2, and Theorem 13)
- 7) $\frac{BC}{AB} = \frac{BD}{BC}$ (by step 3, the definition of similarity, and some rearranging of the ratio)
- 8) $BD = \frac{(BC)^2}{AB}$ (by step 4 and algebra)
- 9) $AB = AD + BD$ (by ruler postulate)
- 10) $AB = \frac{(AC)^2}{AB} + \frac{(BC)^2}{AB}$ (substituting 5 and 8 into 9)
- 11) $(AC)^2 + (BC)^2 = (AB)^2$ (multiplied both sides of 10 by AB)

End of proof

In the exercises, you will be asked to produce counterexamples to show that the Pythagorean theorem does not hold in hyperbolic or spherical geometry.

Theorem 15 (converse of the Pythagorean Theorem, for Euclidean geometry): In Euclidean geometry, if the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the angle opposite the third side is a right angle.

Proof

- 1) In Euclidean geometry, let $\triangle ABC$ be a triangle such that $(AC)^2 + (BC)^2 = (AB)^2$. (We must show that $\sphericalangle C$ is a right angle.)
- 2) There exists a right angle $\sphericalangle GFH$. (absolute geometry axioms and theorems)
- 3) There exists a point D on \overline{FG} such that $\overline{DF} \cong \overline{AC}$. (absolute geometry axioms and theorems)
- 4) There exists a point E on \overline{FH} such that $\overline{EF} \cong \overline{BC}$. (absolute geometry axioms and theorems)
- 5) Triangle $\triangle DEF$ is a right triangle with right angle at F . (by steps 2, 3, 4, and definition of right triangle.)

$$6) (DF)^2 + (EF)^2 = (DE)^2 \text{ (Pythagorean theorem applied to } \triangle DEF \text{.)}$$

$$7) DE = AB \text{ (by 3, 4, 6, and algebra)}$$

$$8) \triangle ABC \cong \triangle DEF \text{ (by SSS theorem of absolute geometry)}$$

$$9) \sphericalangle C \cong \sphericalangle F \text{ (by step 8 and CPCTC)}$$

$$10) \sphericalangle C \text{ is a right angle (by steps 2 and 9)}$$

End of proof

In the exercises, you will be asked to produce counterexamples to show that the converse of the Pythagorean theorem does not hold in hyperbolic or spherical geometry.

Our attention now turns to the area of triangles in Euclidean geometry. We start with an official definition of *base* and *height*,

Definition 9 *base* and *height* for a triangle in absolute geometry

- words: *base* of a triangle and the corresponding *height*
- meaning: the word *base* is simply a title given to one of the three sides of a triangle. Any of the three sides may be the one chosen to be called the *base*. Once a side has been designated as the *base*, the corresponding *height* is the length of the altitude drawn from the opposite vertex.

Observe that the concepts of *base* and *height* apply to triangles in both Euclidean and hyperbolic geometry. The following important theorem, however, only applies in Euclidean geometry.

Theorem 16 (In Euclidean triangles, *base* · *height* is independent of choice of base) In Euclidean geometry, the value obtained for *base* · *height* of a triangle does not depend on which of the triangle's three sides is chosen as the base.

Proof:

1) In Euclidean geometry, let $\triangle ABC$ be a triangle with altitudes \overline{AD} , \overline{BE} , and \overline{CF} . (We must show that $AB \cdot CF = BC \cdot AD = CA \cdot BE$.)

Show that $BC \cdot AD = CA \cdot BE$

2) Either point D lies on segment \overline{BC} , or it lies somewhere outside of \overline{BC} on line \overline{BC} . Similarly, either point E lies on segment \overline{AC} , or it lies somewhere outside of \overline{AC} on line \overline{AC} . That means that there are four cases to consider.

Case 1: Suppose that point D lies on segment \overline{BC} and point E lies on segment \overline{AC} .

3) $\sphericalangle ACD \cong \sphericalangle BCE$ (by reflexivity)

4) $\sphericalangle CDA \cong \sphericalangle CEB$ (right angles)

5) $\triangle CDA \sim \triangle CEB$ (by Theorem 11, AA similarity theorem for Euclidean geometry)

$$6) \frac{AD}{AC} = \frac{\text{leg}}{\text{hypotenuse}} = \frac{BE}{BC} \text{ (by proportionality of sides in similar triangles)}$$

7) $BC \cdot AD = CA \cdot BE$ (cross multiplied)

Case 2: Suppose that point D lies on segment \overline{BC} and point E lies outside segment \overline{AC} .

8) $\sphericalangle ACD \cong \sphericalangle BCE$ (by reflexivity)

9) $\sphericalangle CDA \cong \sphericalangle CEB$ (right angles)

10) $\triangle CDA \sim \triangle CEB$ (by Theorem 11, AA similarity theorem for Euclidean geometry)

11) $\frac{AD}{AC} = \frac{\text{leg}}{\text{hypotenuse}} = \frac{BE}{BC}$ (by proportionality of sides in similar triangles)

12) $BC \cdot AD = CA \cdot BE$ (cross multiplied)

Case 3: Suppose that point D lies outside segment \overline{BC} and point E lies on segment \overline{AC} .

13) $\sphericalangle ACD \cong \sphericalangle BCE$ (by reflexivity)

14) $\sphericalangle CDA \cong \sphericalangle CEB$ (right angles)

15) $\triangle CDA \sim \triangle CEB$ (by Theorem 11, AA similarity theorem for Euclidean geometry)

16) $\frac{AD}{AC} = \frac{\text{leg}}{\text{hypotenuse}} = \frac{BE}{BC}$ (by proportionality of sides in similar triangles)

17) $BC \cdot AD = CA \cdot BE$ (cross multiplied)

Case 4: Suppose that point D lies outside segment \overline{BC} and point E lies outside segment \overline{AC} .

18) $\sphericalangle ACD \cong \sphericalangle BCE$ (because they are vertical angles)

19) $\sphericalangle CDA \cong \sphericalangle CEB$ (right angles)

20) $\triangle CDA \sim \triangle CEB$ (by Theorem 11, AA similarity theorem for Euclidean geometry)

21) $\frac{AD}{AC} = \frac{\text{leg}}{\text{hypotenuse}} = \frac{BE}{BC}$ (by proportionality of sides in similar triangles)

22) $BC \cdot AD = CA \cdot BE$ (cross multiplied)

23) We see that in all four cases, $BC \cdot AD = CA \cdot BE$. Therefore, we conclude that $BC \cdot AD = CA \cdot BE$.

Show that $AB \cdot CF = BC \cdot AD$.

24) Repeat the analysis done in steps (2)-(23) with letters A, B, C, D, E , replaced by C, A, B, F, D . The result will be $AB \cdot CF = BC \cdot AD$.

Combine steps (23) and (24)

25) $AB \cdot CF = BC \cdot AD = CA \cdot BE$ by steps (23), (24), and transitivity

End of proof

Remarks: In high school geometry, you undoubtedly encountered the formula for the area of a right triangle: $A = \frac{bh}{2}$, where b and is the length of the *base* of the triangle, and h is the triangle's *height*.

The above Theorem 16 tells us that for a given triangle, the value obtained for the area will not depend on the choice of which of the three sides is to play the role of the *base*. (In the jargon of functions, we would say that the formula for area is “well-defined”.) Also, observe that the formula $A = \frac{bh}{2}$ does not resemble the formula you have learned for the area of a triangle in hyperbolic geometry: $\text{area} = \pi - \text{angle sum}$. We saw earlier that the concepts of *base* and *height* do apply to hyperbolic triangles, but we have not seen a proof that in hyperbolic triangles, the value of *base* · *height* does not depend on the choice of base. It is natural to wonder if there are triangles in hyperbolic geometry such that *base* · *height* does depend on the choice of base. If there are such triangles, then that would explain why nobody ever seems to talk about computing *base* · *height* for a hyperbolic triangle. You will explore this in the exercises.

3.2. Theorems that will be proven in the exercises

Theorem 17 (Hypotenuse-Leg Theorem in absolute geometry) In absolute geometry, if the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the two triangles are congruent.

Theorem 18 (For similar triangles in Euclidean geometry, length proportionality extends to altitudes.)
 In Euclidean geometry, if triangle $\triangle ABC$ is similar to triangle $\triangle DEF$, then

$$\frac{\text{length of any altitude of } \triangle ABC}{\text{length of the corresponding altitude of } \triangle DEF} = \frac{\text{length of any side of } \triangle ABC}{\text{length of the corresponding side of } \triangle DEF}.$$

Theorem 19 In Euclidean Geometry, if $\triangle ABC$ is a triangle and B', C' are points such that $A - B' - B$, $A - C' - C$, and $\sphericalangle ABC \cong \sphericalangle AB'C'$, then $\text{Area}(\triangle ABC') = \text{Area}(\triangle AB'C)$, and

$$\frac{AB'}{AB} = \frac{AC'}{AC}.$$

Theorem 20 In Euclidean geometry, if $\square ABDE$, $\square BCFD$, $\square CAHI$ are parallelograms on the outside of $\triangle ABC$, and $H - A - E$ and $\overline{HA} \cong \overline{AE}$, then $\text{Area}(\square ABDE) + \text{Area}(\square BCFD) = \text{Area}(\square CAHI)$.

Theorem 21 (a generalized Pythagorean Theorem) In Euclidean geometry, if $\square ABDE$, $\square BCFG$, $\square CAHI$ are parallelograms on the outside of $\triangle ABC$ such that $\overline{AH} \parallel \overline{BP}$ and $\overline{AH} \cong \overline{BP}$, where P is the intersection of lines \overline{DE} and \overline{FG} , then $\text{Area}(\square ABDE) + \text{Area}(\square BCFG) = \text{Area}(\square CAHI)$.

3.3. Exercises

[1] Discuss why the concept of altitude is problematic in spherical geometry. (Draw pictures to illustrate.)

[2] Prove Theorem 17. Hint: In absolute geometry, let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\sphericalangle C$ a right angle, $\sphericalangle F$ is a right angle, $\overline{AB} \cong \overline{DE}$, and $\overline{AC} \cong \overline{DF}$. (You must show that $\triangle ABC \cong \triangle DEF$.) Introduce a point G such that $E - F - G$ and $\overline{GF} \cong \overline{BC}$. Then show that $\triangle DGF \cong \triangle ABC$. Then show that $\triangle DGF \cong \triangle DEF$. (Justify all steps by citing axioms and prior theorems of absolute geometry)

[3] There is no hypotenuse-leg theorem in spherical geometry. Give a counterexample to illustrate.

[4] Prove Theorem 18. Hint: Let G, H, I, J, K , and L be the points at the base of the altitudes from vertices A, B, C, D, E , and F . Step 1: Show that $\frac{AG}{DJ} = \frac{\text{length of any side of } \triangle ABC}{\text{length of the corresponding side of } \triangle DEF}$ is true.

Call this statement (*). To prove that this statement is true, you should prove that it is true in each of the following five cases:

Case 1: $G-B-C$. Case 2: $G=B$. Case 3: $B-G-C$. Case 4: $G=C$. Case 5: $B-C-G$

Step 2: (Generalize to the other altitudes.) Explain what would happen if you were to repeat step 1, with the vertices B, C, A and E, F, D (along with their corresponding altitudes) playing the old roles of vertices A, B, C and D, E, F (along with their corresponding altitudes). (Don't go through all the cases again; just explain what the conclusion would be. The conclusion should be a new statement like statement (*).)

Step 3: (Generalize again) Explain what would happen if you were to repeat step 1, with the vertices C, A, B and F, D, E (along with their corresponding altitudes) playing the old roles of vertices A, B, C and D, E, F (along with their corresponding altitudes). (Don't go through all the cases again; just explain what the conclusion would be. The conclusion should be a new statement like statement (*).) (Justify all steps by citing axioms and prior theorems of absolute or Euclidean geometry.)

[5] Prove Theorem 19 using the following steps. Step 1: prove that $\frac{\text{Area}(\triangle AB'C')}{\text{Area}(\triangle ABC')} = \frac{AB'}{AB}$. Step 2:

prove that $\frac{\text{Area}(\triangle AB'C')}{\text{Area}(\triangle AB'C)} = \frac{AC'}{AC}$. Step 3: prove that $\text{Area}(\triangle BCB') = \text{Area}(\triangle BCC')$. Step 4: prove that

$Area(\triangle ABC') = Area(\triangle AB'C)$. Step 5: prove that $\frac{AB'}{AB} = \frac{AC'}{AC}$. (Justify all steps by citing axioms and prior theorems of absolute or Euclidean geometry.)

[6] Reprove the AAA similarity theorem in Euclidean geometry using area techniques similar to those used in the proof of Theorem 19.

Hint: In Euclidean geometry let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. Your goal is to prove that $\triangle ABC \sim \triangle DEF$. But since you already know that corresponding angles are congruent, you just need to prove that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. Start by proving that $\frac{AB}{DE} = \frac{CA}{FD}$.

(Assume that $AB \neq DE$, because otherwise the triangles are congruent by ASA congruence. Assume further that $AB > DE$, because otherwise we can relabel the triangles, interchanging A, B, C with D, E, F , so that $AB > DE$ will be true.) Let B' be the point such that $A-B'-B$ and $AB' = DE$. Let L be the line such that L contains B' and $L \parallel \overline{BC}$. Using area techniques like those used in the proof of Theorem

19, you should be able to prove that $\frac{AB}{DE} = \frac{CA}{FD}$. The equation just proven can be combined with the earlier assumption $AB > DE$ to give a new inequality $CA > FD$. Now start over. Explain what will happen if you were to repeat all that you just did, but with vertices C, A, B and F, D, E playing the roles of A, B, C and D, E, F . (Justify all steps by citing axioms and prior theorems of absolute or Euclidean geometry.)

[7] Prove Theorem 20. Hint: Show that $polygon(ABCFDE)$, $polygon(ACFE)$, and $polygon(CAHI)$ all have the same area. (Justify all steps by citing axioms and prior theorems of absolute or Euclidean geometry.)

[8] (a) Prove Theorem 21. Hint: Show that lines \overline{AH} and \overline{DE} intersect, and denote by J their point of intersection. Then show that lines \overline{CI} and \overline{FG} intersect, and denote by K their point of intersection. Then show that $Area(\square ABDE) = Area(\square ABPJ)$ and $Area(\square BCFG) = Area(\square BCKP)$. Then apply Theorem 20 to parallelograms $\square ABPJ$, $\square BCKP$, $\square CAHI$ and triangle $\triangle ABC$. Finally, combine results and make a conclusion. (Justify all steps by citing axioms and prior theorems of absolute or Euclidean geometry.)

(b) In what sense is Theorem 21 “more general” than the usual Pythagorean theorem?

[9] (a) Use the picture from exercise 2 in section 2.6 in a counterexample to demonstrate that Theorem 13 could not be a theorem in hyperbolic geometry.

(b) Produce a counterexample to demonstrate that Theorem 13 could not be a theorem in spherical geometry.

[10] (a) Use the same picture from exercise [9a] in a counterexample to demonstrate that there cannot be a Pythagorean theorem in hyperbolic geometry. Hint: to make your work easier, assume that $AB = 1$.

(b) Produce a counterexample to demonstrate that there cannot be a Pythagorean theorem in spherical geometry.

[11] Produce a counterexample to demonstrate that the converse of the Pythagorean theorem could not be a theorem in spherical geometry.

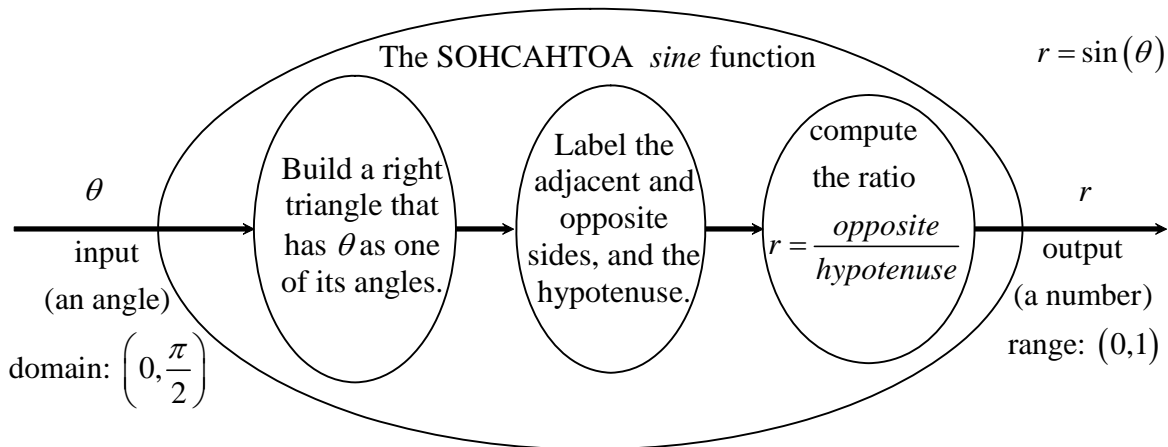
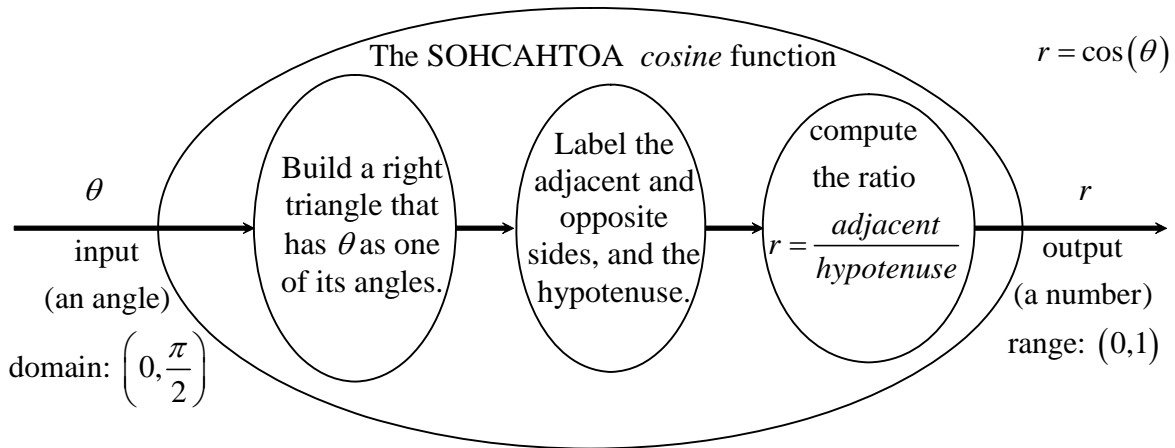
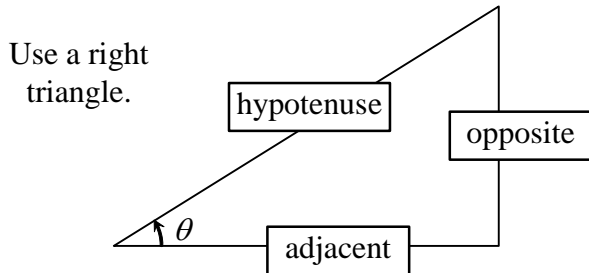
[Challenge Problems] (a) Without using a computer program, produce a counterexample to demonstrate that the converse of the Pythagorean theorem could not be a theorem in hyperbolic geometry.

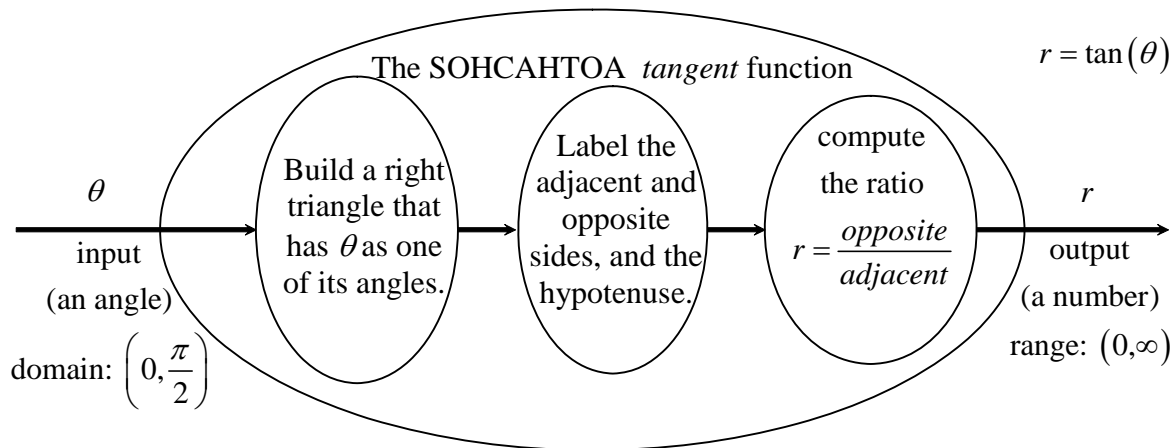
(b) Without using a computer program, produce a counterexample to demonstrate that Theorem 16 could not be a theorem in hyperbolic geometry.

4. Trigonometry Reference

4.1. Three versions of the trig functions

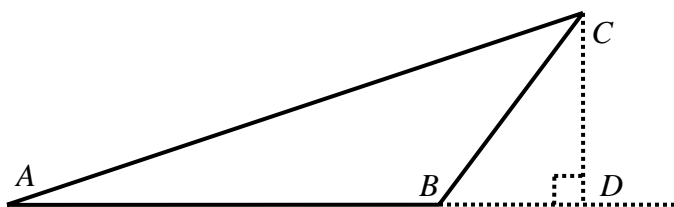
1.1. The SOHCAHTOA trigonometric functions





1.2. The Extended SOHCAHTOA trig functions

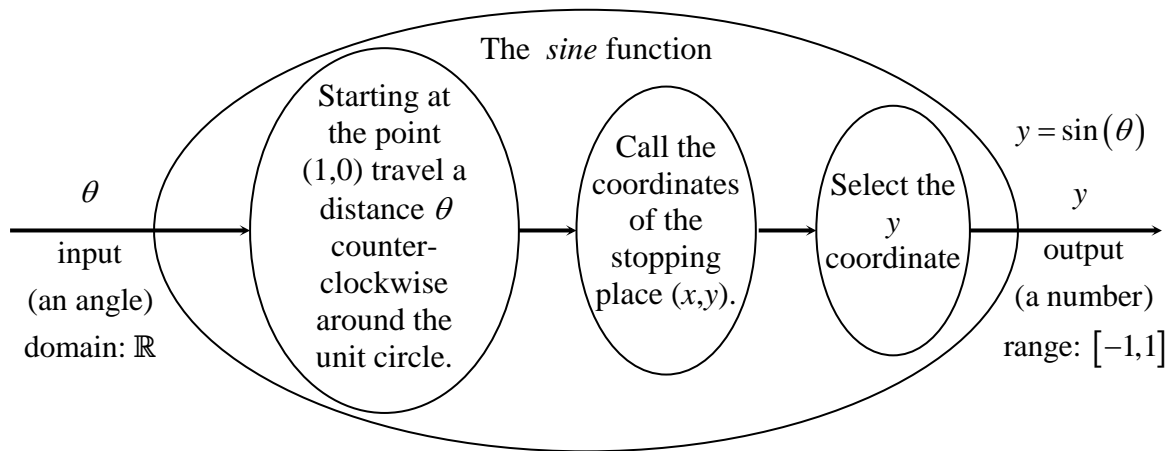
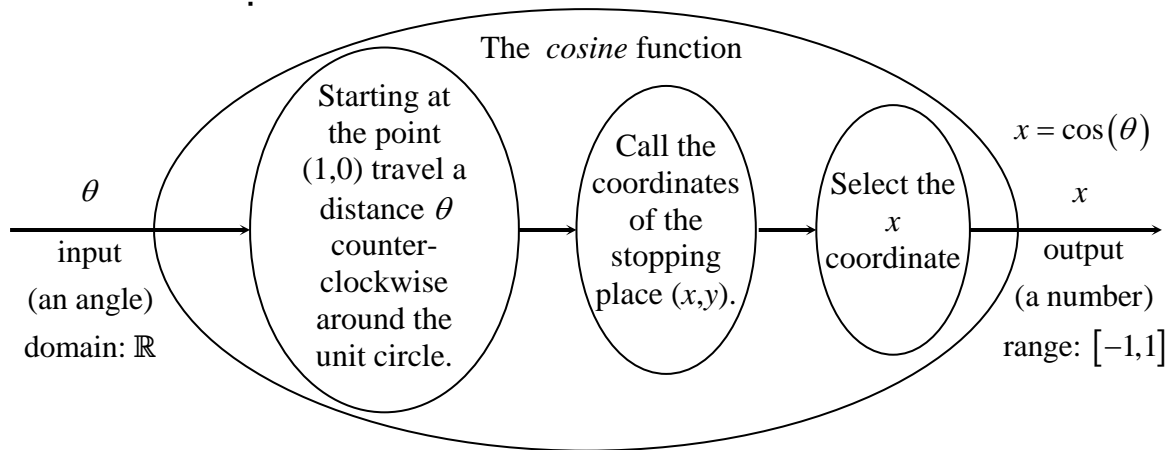
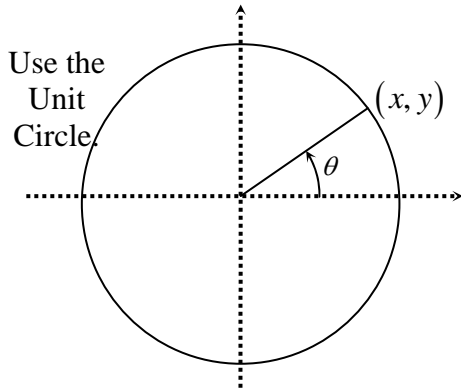
The SOHCAHTOA versions of the trig functions do not apply to $\triangle ABC$ in the figure below, because $\sphericalangle ABC$ is obtuse. For triangles such as this, we can use the *extended SOHCAHTOA* versions of the trig functions.

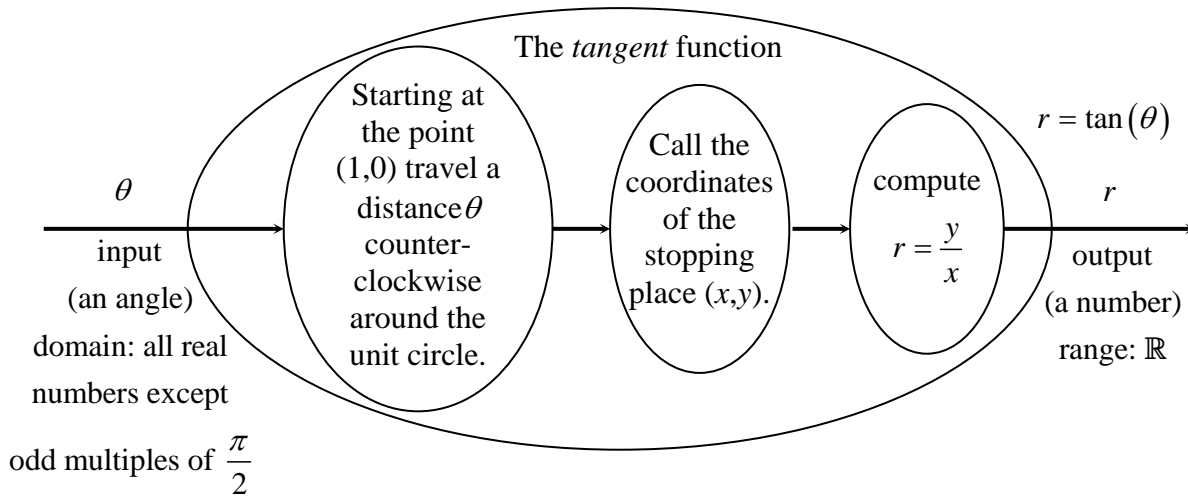


Extended SOHCAHTOA trigonometric functions.	
sine	For $0^\circ < \theta < 90^\circ$ define $\sin(\theta)$ using the SOHCAHTOA process.
	For $\theta = 90^\circ$, define $\sin(\theta) = 1$.
	For $90^\circ < \theta < 180^\circ$, define $\sin(\theta) = \sin(180^\circ - \theta)$.
cosine	For $0^\circ < \theta < 90^\circ$ define $\cos(\theta)$ using the SOHCAHTOA process.
	For $\theta = 90^\circ$, define $\cos(\theta) = 0$.
	For $90^\circ < \theta < 180^\circ$, define $\cos(\theta) = -\cos(180^\circ - \theta)$.
tangent	For $0^\circ < \theta < 90^\circ$ define $\tan(\theta)$ using the SOHCAHTOA process.
	For $\theta = 90^\circ$, $\tan(\theta)$ is undefined.
	For $90^\circ < \theta < 180^\circ$, define $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

1.3. The Unit Circle trig functions

In scientific settings, it is useful to have trig functions whose domains are the set of all real numbers. In these settings, trig functions are defined using the unit circle.





1.4. The *Capital trig functions*

Neither the extended SOHCAHTOA trig functions nor the unit circle trig functions are one-to-one functions. In settings where one needs to consider the inverse trig functions, one must restrict the domain of the trig functions. The result is a new family of trig functions that we could call the *Capital trig functions*. We won't discuss them in this course.

1.5. Summary of the trig functions

Family	Function	Domain	Range	bijjective
SOHCAHTOA	SOHCAHTOA cos	$\left(0, \frac{\pi}{2}\right)$	(0,1)	yes
	SOHCAHTOA sin	$\left(0, \frac{\pi}{2}\right)$	(0,1)	yes
	SOHCAHTOA t	$\left(0, \frac{\pi}{2}\right)$	$(0, \infty)$	yes
extended SOHCAHTOA	extended SOHCAHTOA cos	$(0, \pi)$	(-1,1)	yes
	extended SOHCAHTOA sin	$(0, \pi)$	$(0, 1]$	no
	extended SOHCAHTOA tan	$(0, \pi)$ except $\frac{\pi}{2}$	\mathbb{R} except 0	yes
unit circle	unit circle cos	\mathbb{R}	$[-1, 1]$	no
	unit circle sin	\mathbb{R}	$[-1, 1]$	no
	unit circle tan	\mathbb{R} except odd multiples of $\frac{\pi}{2}$	\mathbb{R}	no
Capital	$Sin()$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$	yes
	$Cos()$	$[0, \pi]$	$[-1, 1]$	yes
	$Tan()$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	\mathbb{R}	yes
Inverses of the Capital trig functions	$Sin^{-1}()$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	yes
	$Cos^{-1}()$	$[-1, 1]$	$[0, \pi]$	yes
	$Tan^{-1}()$	\mathbb{R}	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	yes

4.2. Exercises

[1] Prove the following

Theorem 22 (Law of Sines in Euclidean geometry) In Euclidean geometry, the lengths the sides of any triangle are proportional to the sines of their opposite angles. That is, if $\triangle ABC$ is a triangle and a , b , and c are the lengths of the sides opposite angles $\sphericalangle A$, $\sphericalangle B$, and $\sphericalangle C$, then

$$\frac{\sin(\sphericalangle A)}{a} = \frac{\sin(\sphericalangle B)}{b} = \frac{\sin(\sphericalangle C)}{c}.$$

Hint: Use the extended SOHCAHTOA versions of the trig functions. Do the proof in three steps. In step

1, prove that $\frac{\sin(\sphericalangle A)}{a} = \frac{\sin(\sphericalangle B)}{b}$. Let \overline{CE} be the altitude from vertex C . You will need to consider the

following five cases: Either $E-A-B$, or $E=A$, or $A-E-B$, or $E=B$, or $A-B-E$. In step 2, prove that

$\frac{\sin(\sphericalangle B)}{b} = \frac{\sin(\sphericalangle C)}{c}$ by modifying your work from step 1.

[2] Prove the following

Theorem 23 (Law of Cosines in Euclidean geometry) In Euclidean geometry, if $\triangle ABC$ is a triangle and a , b , and c are the lengths of the sides opposite angles $\sphericalangle A$, $\sphericalangle B$, and $\sphericalangle C$, then

$$c^2 = a^2 + b^2 - 2ab \cos(\sphericalangle C).$$

Hint: Let \overline{AD} be the altitude from vertex C . You will need to consider the following five cases: Either $D-C-B$, or $D=C$, or $C-D-B$, or $D=B$, or $C-B-D$.

5. Review of Unit 1 in preparation for the Midterm Exam

[1] Overview of some of the statements that we studied in Unit 1

The list below contains statements that we have discussed in Unit 1, some with wording slightly changed. For each of the statements, do the following.

- If the statement is implicitly quantified, rewrite the statement using explicit quantifiers.
- Is there a name for the statement?
- Write the negation of the statement.
- Is the statement true in Euclidean geometry? If not, can you give a counterexample?
- Is the statement true in hyperbolic geometry? If not, can you give a counterexample?
- Is the statement true in spherical geometry? If not, can you give a counterexample?

Note that in some instances, there may not be a good answer.

List of statements

- If L and M are lines; and R , S , and T are parallel transversals that intersect line L at points A , B , and C such that $A-B-C$; and R , S , and T intersect line M at points A' , B' and C' ; then $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.
- For any two segments on a line, the ratio of their lengths is the same under every parallel projection.
- If two triangles are similar, then they are congruent.
- If a correspondence between two triangles exists such that corresponding angles are congruent, then the correspondence is a similarity.
- If a correspondence between two triangles exists such that corresponding angles are congruent, then the correspondence is a congruence.
- If a correspondence between two triangles exists such that two pairs of corresponding angles are congruent, then the correspondence is a similarity.
- If a correspondence between two triangles exists such that two pairs of corresponding angles are congruent, then the third pair of corresponding angles is also congruent.
- If a correspondence between two triangles exists such that corresponding sides are proportional, then the correspondence is a similarity.
- In any right triangle, the altitude drawn from the right angle creates two new triangles, each of which is similar to the original triangle.
- The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.
- If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the angle opposite the third side is a right angle.
- the value obtained for $base \cdot height$ of a triangle does not depend on which of the triangle's three sides is chosen as the base.
- If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the two triangles are congruent.
- If triangle $\triangle ABC$ is similar to triangle $\triangle DEF$, then

$$\frac{\text{length of any altitude of } \triangle ABC}{\text{length of the corresponding altitude of } \triangle DEF} = \frac{\text{length of any side of } \triangle ABC}{\text{length of the corresponding side of } \triangle DEF}.$$
- If $\square ABDE$, $\square BCFG$, $\square CAHI$ are parallelograms on the outside of $\triangle ABC$ such that $\overline{AH} \parallel \overline{BP}$ and $\overline{AH} \cong \overline{BP}$, where P is the intersection of lines \overline{DE} and \overline{FG} , then

$$Area(\square ABDE) + Area(\square BCFG) = Area(\square CAHI).$$

[2] Consider Proof Structure

For each of the proofs given in Unit 1 or in the homework solutions, try to do the following.

- a) Make an outline of the proof.
- b) For each section of the outlines, try to understand how the list of proof statements for that section will begin and end.
- c) Proof structure drills with a study partner: pick a section of one of the proof outlines. Present the first and last statements on the list of proof statements for that section to your partner. Ask them to tell you the objective of that portion of the proof.

[3] You have a set of printed proofs for Theorems 1 through 21. The proofs of some of the theorems contain fewer than twelve steps. One the exam, you will be asked to prove one of those theorems. (You will be given a choice of three from which to choose one.)

[4] You will be asked to prove one of the cases of the law of sines or the law of cosines. (There will be some kind of choice available, but it might be that none of the available choices is an easy case.)

[5] You will be given a proof of something, without a picture, and you will be asked to draw a picture to illustrate the proof.