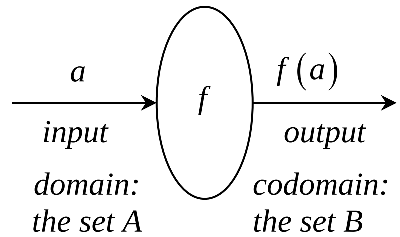
Probability Presented as a Function

Math 450A/550A Summer 2006 (Barsamian)

Definition: *function*

- Symbol: $f : A \to B$
- Spoken: *"f is a function from A to B"*, or *"f maps A to B"*,
- Usage: *A* and *B* are sets, called the *domain* and the *codomain*.
- Meaning: *f* is a machine that takes as input an element of set *A* and produces as output an element of set *B*.
- Machine diagram: We will often use a drawing like the one to the right to visualize a function. We call such a drawing a *machine diagram*.



- Other notation: If an element $a \in A$ is used as the input to the function f, then the symbol $f(a)$ is used to denote the corresponding output. We say that the output $f(a) \in B$ is the *image* of the input $a \in A$.

Definition: *power set*

- Symbol: $Power(S)$
- Spoken: "the power set of *S*"
- Usage: *S* is some set.
- Meaning: $Power(S)$ is the set that is the collection of all the subsets of *S*.

Example: If $S = \{a,b\}$, then $Power(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. (Remember that ϕ is the empty set.)

Remark: The power set is a set. Also, for any set *B*, the set $Power(B)$ will always contain ϕ and *B*.

Definition: *probability function for a countable set*

- Words: "*P* is a probability function for *S*."
- Usage: *S* is some countable set. (That is, *S* is finite or countably infinite.)
- Meaning: *P* is a function $P : Power(S) \to \mathbb{R}$ that has the following three properties

Property 1 (nonnegativity): $P(E) \geq 0$ for all $E \in Power(S)$. That is, for all subsets $E \subset S$.

Property 2 (normalization): $P(S) = 1$

Property 3 (additivity): If $E_1, E_2, E_3, \cdots$ is a finite or infinite list of non-intersecting elements of $Power(S)$, then $P(E_1 \cup E_2 \cup E_3 \cup \cdots) = P(E_1) + P(E_2) + P(E_3) + \cdots$.

If we think of set *S* as a *sample space* for some experiment, then the elements $E \in Power(S)$ are *events*, because they are subsets of *S*. So, a probability function accepts an event as input and produces a number as output.

Notice that the definition of *probability function* does not say how the probability function actually works. It only specifies what qualifications a function has to have in order to be entitled to be called a probability function. In general, for a given countable set *S*, it will be possible to define more than one probability function.

For example, let $S = \{a,b\}$. Consider the functions $f : Power(S) \to \mathbb{R}$ and $g : Power(S) \to \mathbb{R}$ defined by the input-output tables at right. Both of these functions meet all of the requirements in the definition of a probability function for *S*. So both can be called probability functions. (If we wanted to, we could rename them P_1 and P_2.)

input E	output $f(E)$
ϕ	0
$\{a\}$	$1/2$
$\{b\}$	$1/2$
$\{a,b\}$	1

input E	output $g(E)$
ϕ	0
$\{a\}$	$1/3$
$\{b\}$	$2/3$
$\{a,b\}$	1