

Random Variables over Countable Sample Spaces

Notes for Sections 3.1 & 3.2

Math 450A/550A Summer 2006 (Barsamian)

1. Terminology and notation for Image, Preimage, and Range

Definition 1 image of an element

- words: b is the image of a under the mapping f .
- usage: $f : A \rightarrow B$ is a function and $a \in A$ is an element of the domain of f .
- meaning in symbols: $f(a) = b$
- meaning in words: when a is used as input to the mapping f , the resulting output is b .
- observation: the image of an element $a \in A$ is an element $b \in B$.

Definition 2 preimage of an element

- words: the preimage of b under the mapping f .
- symbol: $f^{-1}(b)$
- usage: $f : A \rightarrow B$ is a function and $b \in B$ is an element of the codomain of f .
- meaning in symbols: $f^{-1}(b) = \{a \in A \text{ such that } f(a) = b\}$
- meaning in words: The preimage of an element $b \in B$ under the mapping f is the set whose elements are all the elements $a \in A$ that have the property that the image of a is b . That is, the preimage is the set of all inputs that will cause an output of b .
- observation: The preimage of an element $b \in B$ is a set $f^{-1}(b) \subset A$.

Example: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$. Then $f^{-1}(4) = \{2, -2\}$, $f^{-1}(0) = \{0\}$, and $f^{-1}(-4) = \emptyset = \{ \}$, the empty set. Note that $f^{-1}(4)$ is a set containing two numbers because the function f is not one-to-one. Similarly, $f^{-1}(-4)$ is the empty set because the function f is not onto.

Definition 3 image of a set

- symbol: $f(E)$
- words: the image of E under the mapping f .
- usage: $f : A \rightarrow B$ is a function and $E \subset A$ is a subset of the domain of f .
- meaning in symbols: $f(E) = \{f(e) \text{ such that } e \in E\}$
- meaning in words: the image of a set E is the set of outputs that result when all of the elements of set E are (individually) used as inputs to the map f .
- observation: The image of a set $E \subset A$ is a set $f(E) \subset B$.

Definition 4 preimage of a set

- words: the preimage of set G under the mapping f .
- symbol: $f^{-1}(G)$
- usage: $f : A \rightarrow B$ is a function and $G \subset B$ is a subset of the codomain of f .
- meaning in symbols: $f^{-1}(G) = \{a \in A \text{ such that } f(a) \in G\}$

- meaning in words: The preimage of a set G under the mapping f is the set whose elements are all $a \in A$ that have the property that the image of a is an element of G .
- observation: The preimage of a set $G \subset B$ is a set $f^{-1}(G) \subset A$.

Definition 5 range of a function

- words: the range of f .
- usage: $f : A \rightarrow B$ is a function
- meaning in symbols: $f(A)$
- meaning in words: the range of function is just the image of the domain.
- observation: The range of a function f is a set $f(A) \subset B$.

Remarks

1. The *range* is not the same thing as the *codomain*. The *codomain* of a function f can be thought of as “the kinds of outputs” that come out of f . The *range* of f can be thought of as the “actual outputs”. For example, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$. The *domain* of f is the set \mathbb{R} of real numbers, meaning that we can feed real numbers into the function as inputs. The *codomain* of f is the set \mathbb{R} of real numbers, but the *range* of f is only the set of non-negative real numbers. That is, the “kinds of outputs” are real numbers, but the “actual outputs” are only the non-negative real numbers. In symbols, we could describe the *range* as $f(\mathbb{R}) = \{y \in \mathbb{R} \text{ such that } y \geq 0\}$.
2. In our textbook, the word *range* is used in the same way that I defined the word in Definition 5. Our book does not use the word *codomain*, but I use it in class and in my handouts.
3. In some books, the word *range* is used where we would use the word *codomain*. (Usually in those books, no name is given to the “actual outputs”.) This happens mostly in lower-level math books. The result is that when you read (or hear) the word *range*, you always have to make sure that you understand which meaning the writer (or speaker) means.

2. Countable Sets

The following definitions were given in class.

Definition 6 finite set

- words: S is a finite set
- meaning: The elements of S can be put in a list that ends. That is $S = \{s_1, s_2, s_3, \dots, s_k\}$.
- additional terminology: The symbols $|S|$ and $n(S)$ are used to denote the number of elements in set S . This number is also called the *cardinality* of S .

For example, $S = \{a, b, c, d, e\}$ is a finite set, with $s_1 = a$, etc. Note that $|S| = 5$.

Definition 7 countably infinite set

- words: S is a countably infinite set
- meaning: The elements of S can be put in a list that has a beginning, but never ends. That is $S = \{s_1, s_2, s_3, \dots\}$.

Example 1: The set of positive integers \mathbb{Z}^+ is countably infinite, because we can list the elements of the set: $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

Example 2: The set of integers \mathbb{Z} is countably infinite, but the listing must be done carefully. The clearest description of the set is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, but in this description, the list does not have a beginning or an end. The description $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ gives a list that has a beginning but no end. So it is this second description that shows explicitly that \mathbb{Z} is countably infinite.

Example 3: The set of rational numbers \mathbb{Q} is countably infinite. We discussed this in class.

Definition 8 countable set

- words: S is a countable set
- meaning: The set S is either finite or countably infinite.
- additional terminology: our book uses the words *discrete set* and *countable set* interchangeably.

Definition 9 uncountable set

- words: S is an uncountable set
- meaning: The set S is neither finite nor countably infinite. In other words, no list—not a finite list and not even an unending list—can ever contain all of the elements of set S .

Example 4: The set of real numbers \mathbb{R} is uncountable.

3. Countability of domains, ranges, and codomains.

If a function $f : A \rightarrow B$ has a countable domain A , then the range $f(A)$ will be countable as well. To see why, observe that we can make a list of the elements of $f(A)$ in the following way:

list of elements of A	list of elements of $f(A)$
a_1	$f(a_1)$
a_2	$f(a_2)$
a_3	$f(a_3)$
\vdots	\vdots

For example, consider the function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. The domain is \mathbb{Z}^+ , so only positive integers will be used as inputs. We can list the inputs and their corresponding outputs.

list of elements of \mathbb{Z}^+	list of elements of $f(\mathbb{Z}^+)$
1	1
2	4
3	9
\vdots	\vdots

Notice that in this example, the codomain was said to be the set \mathbb{R} , which is uncountable. But the actual set of outputs—the *range* of the function—is countable because the domain is countable.

4. Random Variable on a Countable Sample Space

Recall that in a previous handout, you saw the definition of a *probability function for a countable set S* . It is important to remember that the definition stated there only works when the set S is countable. Our

book sometimes uses the expression *probability measure*, but the expression is not defined. A probability measure is the same thing as a probability function. So far in our book, we have only encountered probability functions for countable sets. That is, *probability measures for countable sets*.

The following definition is meant to convey the same concepts as Definition 3.1 in the book.

Definition 10 random variable on a countable sample space

- words: X is a random variable over S .
- usage: S is some countable sample space with a probability function.
- meaning in symbols: $X : S \rightarrow \mathbb{R}$
- meaning in words: X is a function whose domain is the set S and whose codomain is the set \mathbb{R} of real numbers.

The inputs to X are elements of the sample space S . Remember that such elements $s \in S$ are called *outcomes*. So an *input* to X is something that is called an *outcome*. You should find this confusing. The *outputs* of X will be *real numbers*. The symbol $X(s)$ stands for the real number output that results when the outcome $s \in S$ is used as input to the function X .

The range of X is denoted by the symbol $X(S)$. It is the set of actual outputs. Because the domain is countable, the range will be countable, even though the codomain is the uncountable set \mathbb{R} . Remember that this can be justified by thinking of a list of inputs and their corresponding outputs.

list of elements of S	list of elements of $X(S)$
s_1	$X(s_1)$
s_2	$X(s_2)$
s_3	$X(s_3)$
\vdots	\vdots

The book refers to this issue on page 72. It says that the name *discrete random variable* is given to any random variable whose *range* is countable. The underlying issue in this paragraph of the book is not stated as well as it might be. The point is that one is often interested in knowing whether the domain of a random variable is countable or uncountable, and also whether the range is countable or uncountable. Here are the possibilities, along with the terminology used for them:

domain of X	range of X	terminology for X
countable	countable	discrete random variable on a discrete sample space
countable	uncountable	This combination cannot happen
uncountable	countable	discrete random variable on a continuous sample space
uncountable	uncountable	continuous random variable on a continuous sample space

At the bottom of page 70 in the textbook is a rather confusing paragraph that is worth clarifying.

“...Thus $X = 9$ is to be interpreted as the set of elements of S for which the total is 9 and, more generally, $X = x$ is to be interpreted as the set of elements of the sample space for which the random variable X takes on the value of x ...”

The author is saying that the symbol $X = 9$ is used to denote the set of inputs that will cause an output of 9, and the symbol $X = x$ is used to denote the set of inputs that will cause an output of x . In other words, the symbol $X = 9$ is used to denote the *preimage of 9*, and the symbol $X = x$ is used to denote

the *preimage* of x . Using the standard preimage notation introduced earlier, we could say that the symbol $X = 9$ means the same thing as the more common symbol $X^{-1}(9)$, and the symbol $X = x$ means the same thing as the symbol $X^{-1}(x)$.

Example:

Experiment: Toss a coin four times and record the results as a string of H's and T's.

Sample Space: S is the set of 4-strings over the alphabet $\{H,T\}$.

Probability Function: the function $P : \text{power}(S) \rightarrow \mathbb{R}$ such that each outcome is equally likely

Random Variable: $X : S \rightarrow \mathbb{R}$ defined by $X(s) = \text{number of heads in the string } s$

We can use a table to list the outputs of the functions P and X when outcomes $s \in S$ are used as inputs.

outcome $s \in S$	$P(s)$	$X(s)$
TTTT	1/16	0
TTTH	1/16	1
TTHT	1/16	1
TTHH	1/16	2
THTT	1/16	1
THTH	1/16	2
THHT	1/16	2
THHH	1/16	3
HTTT	1/16	1
HTTH	1/16	2
HTHT	1/16	2
HTHH	1/16	3
HHTT	1/16	2
HHTH	1/16	3
HHHT	1/16	3
HHHH	1/16	4

From the table, we can see that the range of the function X is the finite set $\{0,1,2,3,4\}$. In symbols, we could write $X(S) = \{0,1,2,3,4\}$.

5. Probability Distributions

Definition 11 probability distribution of a random variable on a countable sample space

- words: f is the probability distribution of X
- usage: X is a random variable over some countable sample space S with a probability function P .
- meaning in symbols: f is the function $f : X(S) \rightarrow \mathbb{R}$ defined by $f(x) = P(X^{-1}(x))$
- meaning in words: f is a function whose domain is the set that is the range of X , and whose codomain is the set of real numbers, and whose outputs are obtained in the following way: when an element x in the range of X is used as input to f , the corresponding output of f is the real number $f(x) = P(X^{-1}(x))$.

Example: Consider again the example of a coin tossed four times, with random variable X being the number of heads obtained. We saw that the range of the function X is the finite set $X(S) = \{0, 1, 2, 3, 4\}$. We can use a table to figure out how the function f that is the probability distribution of X must work. To describe how the function f works, we must consider every element of the domain of f , that is every element $x \in X(S)$, and determine the resulting output $f(x)$.

element $x \in X(S)$	$X^{-1}(x)$	$P(X^{-1}(x))$	$f(x)$
0	$\{TTTT\}$	1/16	1/16
1	$\{TTTH, TTHT, THTT, HTTT\}$	4/16	4/16
2	$\{TTHH, THTH, THHT, HTTH, HTHT, HHTT\}$	6/16	6/16
3	$\{THHH, HTHH, HHTH, HHHT\}$	4/16	4/16
4	$\{HHHH\}$	1/16	1/16

6. Mapping Diagrams for Probability Distributions

In order to better understand what probability distributions do, it is useful to consider what they do not do. In particular, consider the following two processes that can be used to obtain a real number from an element $s \in S$ of the sample space S .

Process #1: Feed s into the random variable X to obtain the output $X(s)$. This output is an element of the range of X . That is, $X(s) \in X(S)$. Then, feed $X(s)$ into the probability distribution f to obtain the real number output $f(X(s))$.

Process #2: Feed s into the probability function P to obtain the real number output $P(s)$. (If you want to be picky, you could say that we feed $\{s\}$ into P to obtain $P(\{s\})$.)

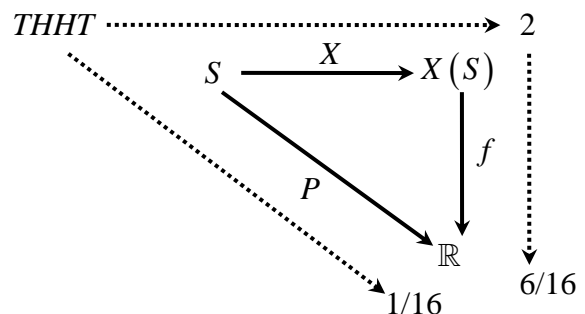
Let's try these processes with the element $s = THHT \in S$ from the example of the coin being tossed four times.

Result of Process #1: $f(X(THHT)) = f(2) = 6/16$

Result of Process #2: $P(THHT) = 1/16$

We see that the results of these two processes are not the same!

The two processes can be illustrated on what is called a *mapping diagram*, shown at right. On the diagram, we see that there are two different paths that lead from the sample space S to the real numbers, \mathbb{R} . The two paths lead to different results.



Now, consider two processes that can be used to obtain a real number from an element $x \in X(S)$ of the range of X .

Process #1: Feed x into the probability distribution f to obtain the real number output $f(x)$.

Process #2: Starting with x , obtain the preimage $X^{-1}(x)$. This will be a subset of the sample

space That is, $X^{-1}(x) \subset S$. Then, feed $X^{-1}(x)$ into the probability function P to obtain the real number output $P(X^{-1}(x))$.

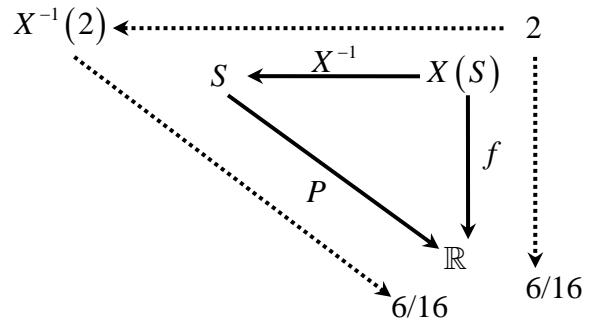
Let's try these processes with the element $2 \in X(S)$ from the example of the 4-coin toss.

Result of Process #1: $f(2) = 6/16$

Result of Process #2: $P(X^{-1}(2)) = P(\{TTHH, THTH, THHT, HTTH, HTHT, HHTT\}) = 6/16$

We see that the results of these two processes are the same!

A mapping diagram for the two processes is shown at right. On the diagram, we see that there are two different paths that lead from the range of X , that is from the set $X(S)$, to the real numbers, \mathbb{R} . The two paths lead to identical results.



7. Cumulative Distributions

The book uses the symbol $X \leq x$ is used to denote the set of inputs that will cause an output less than or equal to x . The set of outputs that are less than or equal to x can be denoted $(-\infty, x]$. So the book's symbol $X \leq x$ is used to denote the preimage of the set $(-\infty, x]$. Using standard preimage notation, we could say that the symbol $X \leq x$ means the same thing as the symbol $X^{-1}((-\infty, x])$.

Definition 12 cumulative distribution of a random variable

- words: F is a cumulative distribution of X .
- usage: X is a random variable over some sample space S .
- meaning in symbols: F is the function $F : X(S) \rightarrow \mathbb{R}$ defined by $F(x) = P(X^{-1}((-\infty, x]))$
- additional terminology: The function F is also called the *distribution function* of X .

Remark: It is not hard to show that an equivalent description of F is $F(x) = \sum_{\substack{t \in f(S) \\ t \leq x}} f(t)$

Example: Consider again the coin toss example.

element $x \in X(S)$	$X^{-1}(x)$	$P(X^{-1}(x))$	$f(x)$	$F(x)$
0	$\{TTTT\}$	1/16	1/16	1/16
1	$\{TTTH, TTHT, THTT, HTTT\}$	4/16	4/16	5/16
2	$\{TTHH, THTH, THHT, HTTH, HTHT, HHTT\}$	6/16	6/16	11/16
3	$\{THHH, HTHH, HHTH, HHHT\}$	4/16	4/16	15/16
4	$\{HHHH\}$	1/16	1/16	1