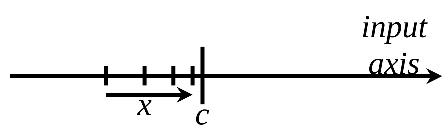
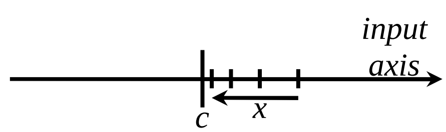
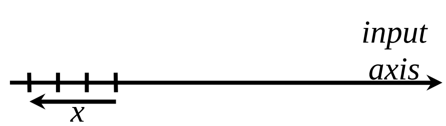
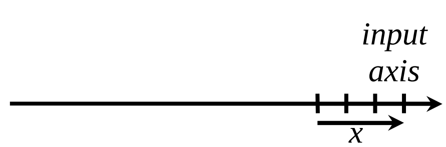
<u>Math 163A Section A03 Handout: Limits and Continuity</u>

In our study of functions this quarter, a lot of our focus has been on single inputs or outputs. For instance, we have asked what will happen if $x = 0$ is fed into a particular function. And we have asked what input x would cause an output of $y = 0$. We have also asked what inputs x would cause the output y to be positive. All of the concepts that we have studied so far have been fairly easy to recognize in a graph, and most of them have been fairly easy to recognize in the corresponding function. The new concepts of *limits* and *continuity* have to do with recognizing certain *trends* in graphs and describing the corresponding behavior of functions. They both deal with graphical behavior that is fairly easy to spot, but the descriptions in terms of functions are difficult.

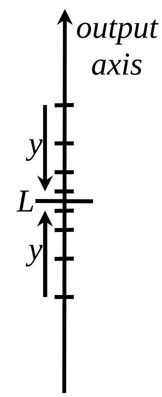
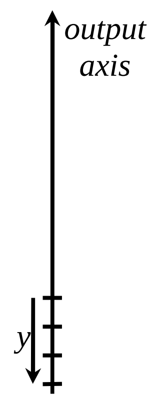
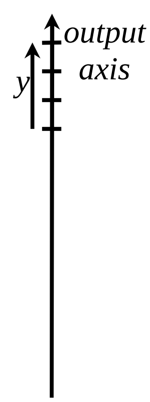
This handout summarizes the content of sections 3.1 and 3.2 of the textbook. But the handout presents the topics in an order very different from that of the textbook. I hope you find this handout useful.

Simple definitions: “One-sided limits” and “limits at infinity”

We start by imagining the graph of a function f, and consider four input trends that are easy to see on such a graph. We introduce a symbol to denote each of the four trends.

Input trend	Picture	Symbol
When the input gets closer & closer to (but not equal to) the real number c, from the left		$\lim\limits_{x \to c^-} f(x)$
When the input gets closer & closer to (but not equal to) the real number c, from the right		$\lim\limits_{x \to c^+} f(x)$
When the input gets more & more negative, without bound (terminology: This is called a *limit at negative infinity*.)		$\lim\limits_{x \to -\infty} f(x)$
When the input gets more & more positive, without bound (terminology: This is called a *limit at infinity*.)		$\lim\limits_{x \to \infty} f(x)$

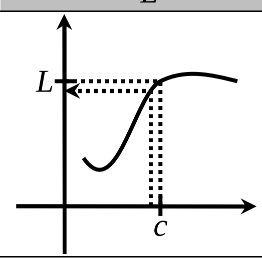
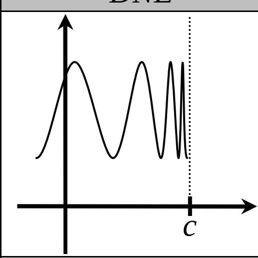
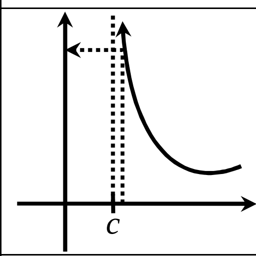
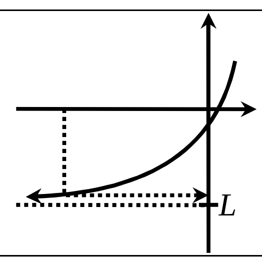
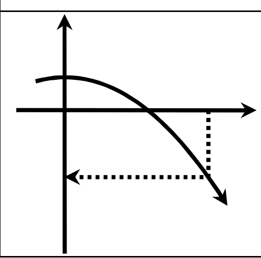
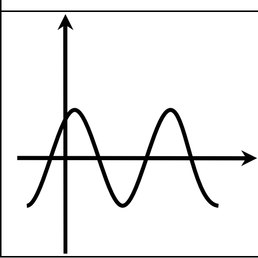
It turns out that for each of the four input trends described above, there are exactly four trends that can occur in the output.

Resulting output trends	Picture	Symbol
the output gets closer & closer to (maybe even equal to) the real number L.		$= L$
the output gets more & more negative, without bound. (terminology: This is called a *limit of negative infinity*.)		$= -\infty$
the output gets more & more positive, without bound. (terminology: This is called a *limit of infinity*.)		$= \infty$
the output does none of the prevous three things.	none of the above pictures	*DNE*

There are sixteen different input trend, output trend pairings. The pairings of the symbols representing the trends results in the sixteen mathematical expressions shown in this table.

	Mathematical expressions	Output Trend			
		$=L$	$=-\infty$	$=\infty$	DNE
Input Trend	$\lim\limits_{x\to c^-} f(x)$	$\lim\limits_{x\to c^-} f(x)=L$	$\lim\limits_{x\to c^-} f(x)=-\infty$	$\lim\limits_{x\to c^-} f(x)=\infty$	$\lim\limits_{x\to c^-} f(x)\ DNE$
	$\lim\limits_{x\to c^+} f(x)$	$\lim\limits_{x\to c^+} f(x)=L$	$\lim\limits_{x\to c^+} f(x)=-\infty$	$\lim\limits_{x\to c^+} f(x)=\infty$	$\lim\limits_{x\to c^+} f(x)\,DNE$
	$\lim\limits_{x\to -\infty} f(x)$	$\lim\limits_{x\to -\infty} f(x)=L$	$\lim\limits_{x\to -\infty} f(x)=-\infty$	$\lim\limits_{x\to -\infty} f(x)=\infty$	$\lim\limits_{x\to -\infty} f(x)\ DNE$
	$\lim\limits_{x\to \infty} f(x)$	$\lim\limits_{x\to \infty} f(x)=L$	$\lim\limits_{x\to \infty} f(x)=-\infty$	$\lim\limits_{x\to \infty} f(x)=\infty$	$\lim\limits_{x\to \infty} f(x)\ DNE$

Each of the sixteen mathematical expressions in the table above can be visualized by a graphical example. Only six of the squares have been filled in.

	Graphical examples	Output Trend			
		$=L$	$=-\infty$	$=\infty$	DNE
Input Trend	$\lim\limits_{x\to c^-} f(x)$				
	$\lim\limits_{x\to c^+} f(x)$				
	$\lim\limits_{x\to -\infty} f(x)$				
	$\lim\limits_{x\to \infty} f(x)$				

And there are examples involving functions given by actual formulas that match the form of each of the sixteen mathematical expressions in the table above.

Formula examples		Output Trend			
		$= L$	$= -\infty$	$= \infty$	*DNE*
Input Trend	$\lim\limits_{x\to c^-} f(x)$	$\lim\limits_{x\to 5^-} \frac{x^2-25}{x-5} = 10$	$\lim\limits_{x\to 5^-} \frac{1}{x-5} = -\infty$	$\lim\limits_{x\to 5^-} \frac{-1}{x-5} = \infty$	$\lim\limits_{x\to 5^-} \left(\sin\left(\frac{1}{x-5}\right)\right)$ *DNE*
	$\lim\limits_{x\to c^+} f(x)$	$\lim\limits_{x\to 5^+} \frac{x^2-25}{x-5} = 10$	$\lim\limits_{x\to 5^+} \frac{-1}{x-5} = -\infty$	$\lim\limits_{x\to 5^+} \frac{1}{x-5} = \infty$	$\lim\limits_{x\to 5^+} \left(\sin\left(\frac{1}{x-5}\right)\right)$ *DNE*
	$\lim\limits_{x\to -\infty} f(x)$	$\lim\limits_{x\to -\infty} \frac{7x}{x+3} = 7$	$\lim\limits_{x\to -\infty} x^3 = -\infty$	$\lim\limits_{x\to -\infty} \left(-x^3\right) = \infty$	$\lim\limits_{x\to -\infty} \left(\sin(x)\right)$ *DNE*
	$\lim\limits_{x\to \infty} f(x)$	$\lim\limits_{x\to \infty} \frac{7x}{x+3} = 7$	$\lim\limits_{x\to \infty} \left(-x^3\right) = -\infty$	$\lim\limits_{x\to \infty} x^3 = \infty$	$\lim\limits_{x\to \infty} \left(\sin(x)\right)$ *DNE*

Compound Definitions: "limit" and "continuous"

The following two definitions are more complicated than the simple limits we discussed above. These new definitions involve combinations of previously defined concepts. For that reason, we could refer to them as *compound definitions*.

Definition of *Limit*

- words: the limit, as x approaches c, of $f(x)$, exits
- symbols: $\lim\limits_{x\to c} f(x)$ exits
- meaning: the function passes this three-part test
  - (a) $\lim\limits_{x\to c^-} f(x)$ must exist.
  - (b) $\lim\limits_{x\to c^+} f(x)$ must exist
  - (c) The values of the limits in test (a) and (b) must match
- In symbols: If $\lim\limits_{x\to c^-} f(x) = *$ and $\lim\limits_{x\to c^+} f(x) = *$, then we write $\lim\limits_{x\to c} f(x) = *$. In this sentence, the asterisk symbol $*$ can be either a real number L (in all three places), or the infinity symbol ∞ (in all three places), or the negative infinity symbol $-\infty$ (in all three places).

Definition of *continuous at a particular x-value*

- words: " f is continuous at $x = c$"
- meaning: the function passes this three-part test
  - (1) $f(c)$ must exist.
  - (2) $\lim\limits_{x\to c} f(x)$ must exist
    - (2a) $\lim\limits_{x\to c^-} f(x)$ must exist.
    - (2b) $\lim\limits_{x\to c^+} f(x)$ must exist
    - (2c) The values of the limits in test (a) and (b) must match
  - (3) The values in tests (1) and (2) must match
- In symbols: If $f(c) = L$ and $\lim\limits_{x\to c} f(x) = L$, then we say that " f is continuous at $x = c$". Note that the symbol L must represent a real number, because it is an output of a function.