Solutions to Geometry Math 330B/539 (Barsamian) In-Class Exam 1
Friday, January 30, 2009

[1] (problem 4.1#7, done in class)
In triangle(ABC), point D is the midpoint of side(AB) and point E is the midpoint of side(AC).
(a) Using similarity, prove the Triangle Midsegment Theorem:
segment(BC) || segment(DE) and BC = 2DE.

Solution:
Show that triangle(BAC) is similar to triangle(DAE)
- *Observe that angle(BAC) is congruent to itself.*
- *Observe that BA/DA = 2 = CA/EA, because points D and E are midpoints.*
- *Therefore, triangle(BAC) ~ triangle(DAE) by ASA similarity.*

Use the similarity result to show that BC = 2DE.
- *We know that BC/DE = 2 because similar triangles have corresponding sides that are proportional.*
- *Conclude that BC = 2DE.*

Use the similarity result to show that segment(BC) is parallel to segment(DE)
- *We know that angle(ABC) is congruent to angle(ADE) because similar triangles have corresponding angles that are congruent.*
- *Therefore, line(BC) is parallel to line(DE) by the converse of the corresponding angle theorem.*

End of proof

(b) Medians BE and CD meet at point F. Using similarity, prove that BF = 2FE and CF = 2FD.

Solution
Show that triangle(CBF) is similar to triangle(DEF)
- *Angle(CBF) is congruent to angle(DEF). (We know from part (a) that BC ||DE. Angle(CBF) and angle(DEF) are alternate interior angles. The Converse of the Alternate Interior Angle Theorem says that they are congruent.)*
- *Angle(BFC) is congruent to (EFD) because they are vertical angles.*
- *Therefore, triangle(CBF) ~ triangle(DEF) by AA similarity.*

Use the similarity result
- *BF/EF = CF/DF = BC/DE because similar triangles have corresponding sides that are proportional.*
- *But we know from part (a) that BC/DE = 2.*
- *Therefore, BF/EF = 2 and CF/DF = 2. Therefore, BF = 2EF and CF = 2DF.*

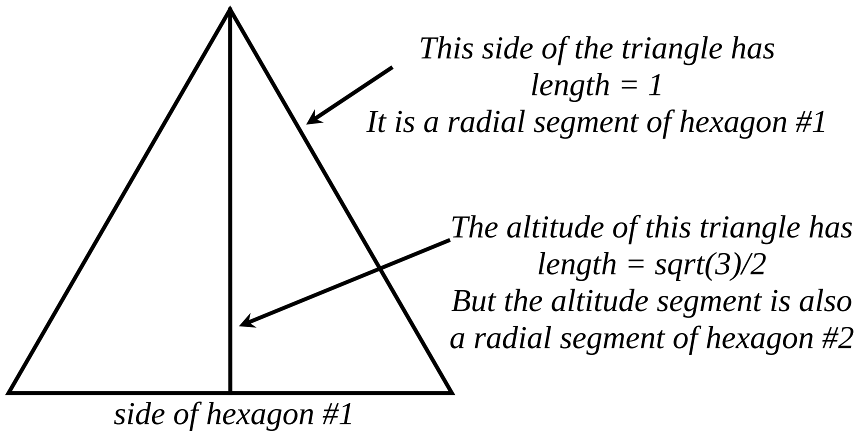
End of proof.

[2] (suggested problem 4.2#2)
In triangle(ABC), angle C is a right angle. D is a point on side(BC) such that ray(AD) bisects angle(A). Which segment is longer: BD or CD? Prove your answer.

Solution
- *BD/CD = BA/CA by Theorem 4.7 (side splitting) applied to triangle(ABC) and bisector ray(AD).*
- *But BA > CA because BA is the hypotenuse and CA is a leg in right triangle(ABC).*
- *Therefore, BA/CA > 1.*
- *Therfeore, BD/CD > 1.*
- *Therefore, BD > CD.*

[3] (part of assigned problem 4.3#5)
A hexagon #1 has sides 1 unit long. A hexagon #2 is inscribed by joining the midpoints of the sides of the first hexagon. What is ratio area(hexagon #1) / area(hexagon #2). Show your work and explain.

Solution:
Consider one of the six equilateral triangular wedges that make up hexagon #1. This wedge will have three sides of length 1. One of the sides of this triangle is a side of hexagon #1. The other two sides of the triangle are radial segments of hexagon #1. An altitude segment of this triangle will be a radial segment of hexagon #1. But the altitude segment will have length = sqrt(3)/2 because the altitude segment is the long leg of a 30-60-90 triangle that has hypotenuse of length 1. See the figure below.



So (length of radius of hex #1)/(length of radius of hex #1) = 1/(sqrt(3)/2) = 2/sqrt(3).
Therefore, (area of hex #1)/(area of hex #2) = $(2/\mathrm{sqrt}(3))^2$ *= 4/3*

[4] (assigned problem 4.4#1)
In triangle(ABC), m(angle(A)) = 36, and m(angle(B)) = m(angle(C)) = 72. Prove that the ratio of lengths BA/BC is equal to φ, the golden ratio.
Hints:

- Let BC = 1. Then your job is to show that BA is equal to φ.
- Let D be a point on AB such that ray(CD) bisects angle(C). Use similarity and side splitting.

Solution (This solution is slightly different from the solution to problem 4.4#1 that I included in the Homework #3 Solutions. I think this solution may be easier.)
(Draw a picture as you read this proof)
Setup

- Suppose that triangle(ABC) has a 36 degree angle at A and 72 degree angles at B and C. Suppose further that BC = 1.
- Let BA = x. Our goal is to find BA/BC. Because CB = 1, we can find the ratio by just finding x.
- Notice that because triangle(ABC) is isosceles, CA = x as well.
- Let D be the point on side(BA) such that ray(CD) bisects angle(C).

Use the Side splitting Theorem

- Apply Theorem 4.7 (Side Splitting) to triangle(ABC) with bisector CD. Result: DA/DB = CA/CB = x.
- Since DA/DB = x, we can multiply both sides by DB to obtain the new equation DA = xDB.

Use segment addition

- From segment addition, we know that DB + DA = x.
- Substituting in DA = xDB, we obtain the new equation DB + xDB = x.
- Factoring this equation, we obtain the new equation DB(1 + x) = x.
- Dividing, we obtain the new equation DB = x/(1 + x). Call this equation *

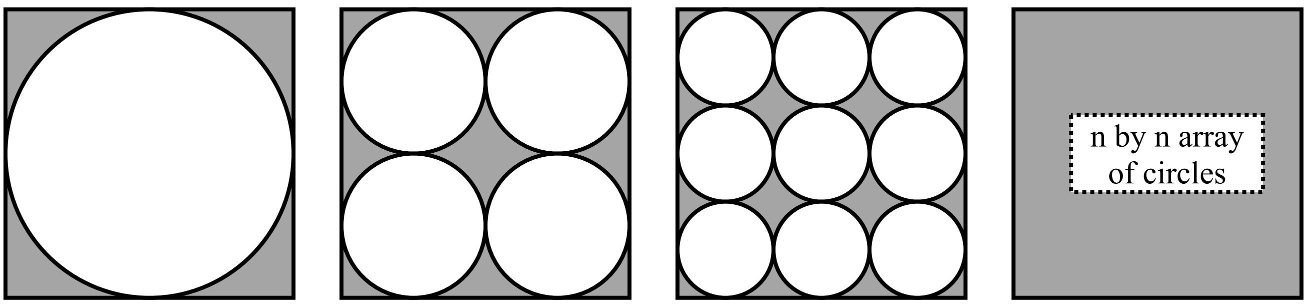
Use fact that the small triangle is similar to the large one

- Note that measure(angle(BCD)) = 36 because ray CD bisects angle(C).
- Observe that triangle(ABC) is similar to triangle(CDB) by AA similarity.
- Therefore CA/CB = CB/BD. That is $x = 1/BD$.
- This tells us that BD = $1/x$. Call this equation **

Combine results

- Combining equation* and equation **, we obtain the new equation $x/(1 + x) = 1/x$.
- Multiplying both sides by $x(1 + x)$ and cancelling, we obtain the new equation $x^2. = x + 1$.
- Subtracting $1 + x$ from both sides of this equation, we obtain the new equation $x^2 - x - 1 = 0$.
- We see that x satisfies the same equation that φ satisfies. We also know that x is positive. Therefore, x must be the same number as φ.

[5] (suggested problem 4.5#3)
In each picture, the square has sides of length 1. Find the shaded areas. Show your work clearly.



picture (a) picture (b) picture (c) picture (d)

Solution:

(a) Shaded area $=$ area of square - area of 1 circle of radius $r = 1/2$
$= 1 - \pi(1/2)^2$
$= 1 - \pi/4$

(b) Shaded area $=$ area of square - area of 4 circles of radius $r = 1/4$
$= 1 - 4\pi(1/4)^2$
$= 1 - \pi/4$

(c) Shaded area $=$ area of square - area of 9 circles of radius $r = 1/6$
$= 1 - 9\pi(1/6)^2$
$= 1 - 9\pi/36$
$= 1 - \pi/4$

(d) Shaded area $=$ area of square - area of n^2 circles of radius $r = 1/(2n)$
$= 1 - n^2\pi(1/(2n))^2$
$= 1 - n^2\pi/(4n^2)$
$= 1 - \pi/4$