

Solutions to Geometry Math 330B/539 (Barsamian) In-Class Exam 2

Friday, February 27, 2009

[1] (50 points) Prove that if a map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is distance-preserving then it is also one-to-one and onto.

Solution: This is problem [5.1#8] from Homework #4. See the Homework #4 Solutions.

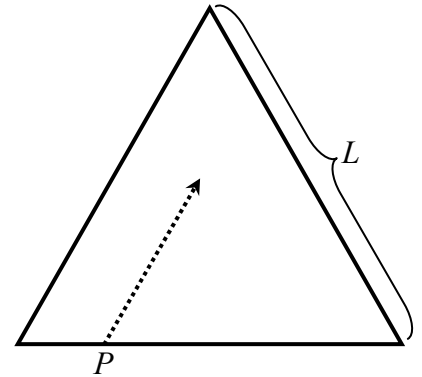
[2] (40 points) (Suggested problem 5.2#3) Three mirrors are arranged in an equilateral triangle with sides of length L . From a point P on one of the mirrors, a beam of light is sent in a direction parallel to one of the other mirrors. Eventually the beam will return to point P and begin repeating its journey along the same path.

(a) Describe the path that the beam of light follows. There are two cases to describe:

- Case 1: point P is a midpoint.
- Case 2: point P is not a midpoint.

(b) In the two cases above, determine the length of the path in terms of L . Explain how you got your answer. (No credit for memorized answers.)

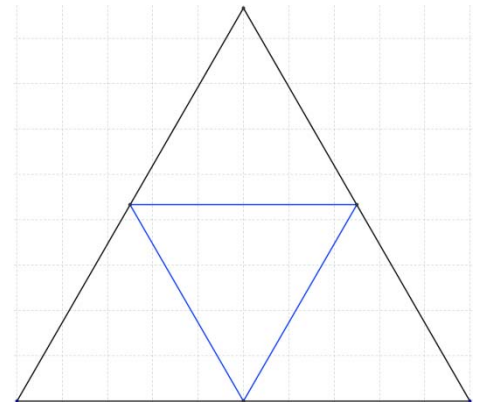
Hint: This problem can be solved using techniques from Section 5.2, but it can also be solved by other methods. The methods used to solve one of your homework problems can be applied.



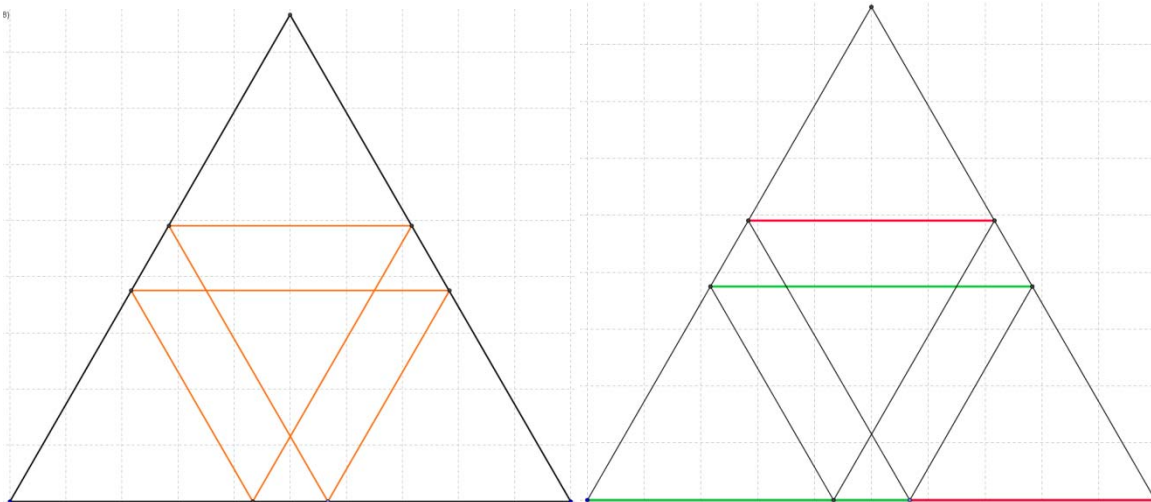
Solution:

Case 1: When point P is a midpoint, the beam of light will bounce twice and return to P . When the beam returns to P , it will bounce and begin retracing its path. The path is an equilateral triangle, as shown in blue in the figure at right. (Note that this is the pedal triangle.) Each of the blue sides is a midsegment of the larger black triangle, so each of the red sides is half as long as the parallel black side. So the blue path length is

$$\text{Blue Path Length} = L/2 + L/2 + L/2 = 3L/2.$$



Case 2: When point P is not a midpoint, the beam of light will bounce five times and return to P . When the beam returns to P , it will bounce and begin retracing its path. The path is shown in orange in the figure at left below.



During our course, I have presented three solutions to the problem of finding the length of the orange path.

Solution 1 (presented in class): Consider the two horizontal portions of the orange path. They are shown in green and red in the figure at right above. Note that they can be moved down to lie on top of the base of the triangle as shown. This tells us that the sum of the lengths of the horizontal portions of the orange path must be equal to L . The same thing can be done for the sloping portions of the orange path. So the orange path length is

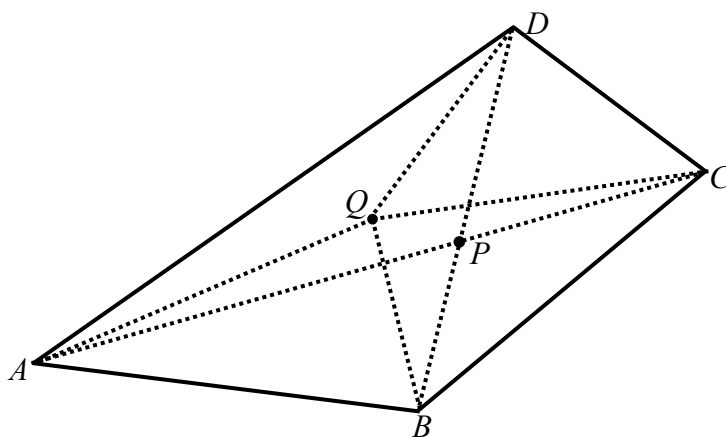
$$\text{Orange Path Length} = L + L + L = 3L.$$

Solution 2: The method that the book outlined in the statement of problem 5.3#3. I provided full details of this solution in my First Proof of 5.3#3 in the Homework 5 Solutions.

Solution 3: The method that I presented in my Second Proof of problem 5.3#3 in the Homework 5 Solutions.

[3] (40 points) (Suggested problem 5.3#5) Given a convex quadrilateral ($ABCD$), prove that the point P at the intersection of the diagonals is the minimum distance point for the quadrilateral. That is, point P is the point from which the sum of the distances to the vertices is minimal.

Hint: Let Q be some other point and compare the sum $QA + QB + QC + QD$ to the sum $PA + PB + PC + PD$ by applying the triangle inequality twice.

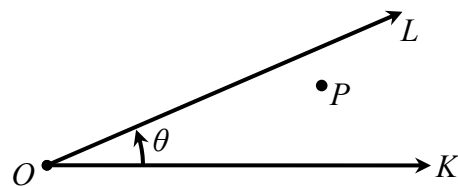


Solution: Let's compute $SUM_P = PA + PB + PC + PD$ and compare it to $SUM_Q = QA + QB + QC + QD$.

$$\begin{aligned} SUM_P &= PA + PB + PC + PD \\ &= (PA + PC) + (PB + PD) \\ &= AC + BD \\ &\leq (AQ + QC) + BD && \text{with equality only if point } Q \text{ lies on diagonal segment}(AC) \\ &\leq (AQ + QC) + (BQ + QD) && \text{with equality only if point } Q \text{ lies on diagonal segment}(BD) \\ &= QA + QB + QC + QD \\ &= SUM_Q \end{aligned}$$

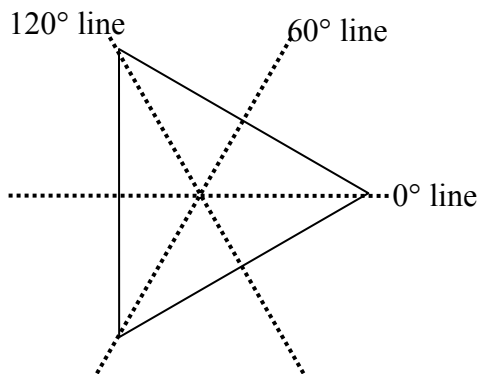
We see that SUM_P will be strictly less than SUM_Q unless point Q lies on diagonal segment(AC) and also lies on diagonal segment(BD). But if point Q lies on diagonal segment(AC) and also lies on diagonal segment(BD), then point Q is actually the same point as point P . Conclude that point P is the minimum distance point.

[5] (40 points) (Suggested problem 6.1#7) Let point P be in the interior of angle θ formed by the lines K and L as shown. Find the points P' and P'' defined by $P \xrightarrow{M_K} P' \xrightarrow{M_L} P''$. Show that in this case, $m(\angle POP'') = 2\theta$. (This shows that if P is in the interior of θ , then P'' can be obtained from P by a rotation $R_{O,2\theta}$.)



Solution: The solution is easy to describe in polar coordinates, assuming that point O is the origin and line K is the x -axis. Suppose that $P = (r, \alpha)$. When P is reflected across line K , the resulting new point is $P' = (r, -\alpha)$. This will be a point below the x -axis (that is, below line K). The counterclockwise angle from ray(OP') to ray(OL) will then be $\alpha + \theta$. When point P' is reflected across line L , the resulting new point is $P'' = (r, (\theta + (\theta + \alpha))) = (r, \alpha + 2\theta)$. Conclude that $M_L \circ M_K = R_{O,2\theta}$.

[6] (40 points) In this problem, you will explore the *dihedral group with eight elements*, D_6 . This is the group of symmetries of an equilateral triangle. If we orient the triangle with its center at the origin, and one of the vertices lying on the positive x -axis, as shown in the figure below, then the elements of this group are as shown in the list to the right.



Elements of the group D_6

R_{0° = identity = id

R_{120° = counterclockwise rotation through 120°

R_{240° = counterclockwise rotation through 240°

M_1 = reflection in the 0° line

M_2 = reflection in the 60° line

M_3 = reflection in the 120° line

(In the figure, you should think of the degree lines as being fixed to the table, while the square is the thing that gets rotated and flipped. For instance, the 0° line is always horizontal, and the 60° line always goes from lower left to upper right, as shown.)

Your task is to fill out the group table below

\circ		transformation to the right of \circ symbol					
		id	R_{120°	R_{240°	M_1	M_2	M_3
transformation to the left of \circ symbol	id						
	R_{120°						
	R_{240°						
	M_1						
	M_2						
	M_3						

Solution:

\circ		transformation to the right of \circ symbol					
		id	R_{120°	R_{240°	M_1	M_2	M_3
transformation to the left of \circ symbol	id	id	R_{120°	R_{240°	M_1	M_2	M_3
	R_{120°	R_{120°	R_{240°	id	M_2	M_3	M_1
	R_{240°	R_{240°	id	R_{120°	M_3	M_1	M_2
	M_1	M_1	M_3	M_2	id	R_{240°	R_{120°
	M_2	M_2	M_1	M_3	R_{120°	id	R_{240°
	M_3	M_3	M_2	M_1	R_{240°	R_{120°	id