Products of four or more reflections

Lemma 1 *Every product $H_P R_v R_u$ can be replaced by a product $R_l R_m$.*

Proof. Let $H_P R_v R_u$ be the motion we wish to analyze. We need to consider three possibilities: the lines u and v intersect, the lines u and v have a common perpendicular or neither.

First suppose that u and v meet at point Q. Let s be the line containing P and Q and let l be a line perpendicular to s passing through P. Note that $H_P = R_l R_s$ and that, as u, v and s all meet at Q, there is a line m such that $R_m = R_s R_v R_u$. It follows that $H_P R_v R_u = (R_l R_s)(R_v R_u) = R_l (R_s R_v R_u) = R_l R_m$.

Now suppose that u and v have a common perpendicular t. Let s be a line perpendicular to t that contains the point P and let l be a line perpendicular to s containing the point P. Note that $H_P = R_l R_s$ and that, as u, v and s are all perpendicular to t, there is a line m perpendicular to t such that $R_m = R_s R_v R_u$. It follows that $H_P R_v R_u = (R_l R_s)(R_v R_u) = R_l (R_s R_v R_u) = R_l R_m$. ■

Lemma 2 *Every product of the form $R_w R_v R_u$ can be replaced by a product of the form $H_P R_l$.*

Proof. Let U be a point on u. Let v' be a line in the pencil formed by w and v that passes through the point U. (This is the join theorem.) Let w' be a line in the pencil such that $R_{w'} = R_{v'} R_v R_w$. Now drop a perpendicular l from U to w'. Observe that l, v' and u are concurrent and hence there is a line a such that $R_a = R_l R_{v'} R_u$. Set $H_P = R_{w'} R_l$ and note that $R_{w'} R_{v'} = R_w R_v$. Hence

$$\begin{aligned} R_w R_v R_u &= R_{w'} R_{v'} R_u \\ &= R_{w'} R_l R_l R_{v'} R_u \\ &= H_P R_l R_{v'} R_u \\ &= H_P R_a. \end{aligned}$$

■

Theorem 3 *Let M be a motion. Then M can be described as the product of three or fewer reflections.*

Proof. First suppose that M has been described as the product of four reflections, i.e., $M = R_4R_3R_2R_1$. Note that there is a half-turn H_P and a reflection R_u such that $H_PR_u = R_4R_3R_2$ and hence $M = H_PR_uR_1$. Now, by the first lemma in this section, there are lines l and m such that $M = R_mR_l$. Let also recall that a reflection can also be described as the product of three reflections, i.e. $R_l = R_lR_mR_m$ for any line m.

We proceed by induction to establish the result for products of five or more reflections. First we make the induction hypothesis: Suppose that M is a motion which has been described as the product of n reflections. If n is odd, then M can be described as the product of three reflections and if n is even then M can be described as the product of two reflections.

It is clear that the induction hypothesis is true when $n = 3$ or $n = 4$.

Now suppose that M is the product of $n + 1$ reflections, i.e., $M = R_{n+1}R_nR_{n-1\ldots}R_4R_3R_2R_1$. First suppose that $n + 1$ is even. Then n is odd and hence$R_nR_{n-1\ldots}R_4R_3R_2R_1$ can be reduced to a product of three reflections, say $R_wR_vR_u$. Thus $M = R_{n+1}R_wR_vR_u$ and hence M can be further reduced to product of two reflections. Now suppose that $n+1$ is odd: then n is even and thus $R_nR_{n-1\ldots}R_4R_3R_2R_1$ can be reduced to a product of two reflections, say R_vR_u. Thus $M = R_{n+1}R_vR_u$, which is the product of three reflections.

Note that we have shown any proper motion can be reduced to the product of two reflections and that any improper motion can be described as a product of three reflections. ■