Math 330B - Exam 2 - Solutions

1. State the following definitions: (3 pts. each)
a. The definition of a glide reflection.
b. The 'vector' definition of a reflection.

Both of these definitions appear in the notes.

2. Circle the best answer. (2 pt. each)
a. In an elliptic plane, a motion (**always**) has one (or more) fixed points. - Each motion is a rotation.

b. In a Euclidean plane, if a motion has exactly one fixed point, then the motion is (**never**) a translation.

c. In a Euclidean plane, if a motion has two intersecting fixed lines, then the motion is (**sometimes**) a reflection - It could be a reflection or a half turn.

d. In a Euclidean plane, if a motion has exactly one fixed line, then the motion is (**always**) a glide reflection.

e. A motion of the form $H_P R_a$ is (**sometimes**) a proper motion. - In the elliptic plane, every motion is a rotation and hence a proper motion.

f. In hyperbolic or Euclidean geometry, if a line is transversal to two parallel lines, then the corresponding angles are (**sometimes**) congruent. - That there are always congruent is equivalent to the Euclidean parallel postulate.

g. In hyperbolic or Euclidean geometry, if a line that is transversal to two lines forms congruent corresponding angles, then the lines are (**always**) parallel.- This is a theorem of absolute geometry.

h. A proof of the Pythagorean theorem (**always**) relies upon the Euclidean Parallel Postulate. - In fact, in metric geometry the Pythagorean theorem is equivalent to the Euclidean parallel postulate.

i. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ corresponds to a (**rotation**). This corresponds to a rotation of 90 degrees.

j. If a geometry satisfies the Ruler Postulate, a line (**always**) contains an infinite number of points. - Since a coordinate system is one-to-one and onto, there will be as many points on the line as there are real numbers.

3. The *law of cosines* asserts that for any triangle $\triangle ABC$ one has that $AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos(\angle C)$. Provide the reason for each step in the following proof. This proof is similar in approach and style to the proof of

the law of sines. Since you do not have an axiom and theorem list for absolute geometry, it is sufficient to give an informal statement of the axiom or theorem you are citing.

Let D be the foot of the perpendicular dropped from B to the line AC.

a. Then one of the following cases must occur: $D-A-C$, $D=A$, $A-D-C$, $D=C$, or $A-C-D$. Why?

This follows from the betweenness axioms. Given three distinct collinear points, one must be between the other two. The cases $D=A$ and $D=C$ cover the situation of only two distinct points.

We will only consider the case $A-C-D$. (The other cases are similar; you may enjoy trying them at home.) above diagram may be useful.

b. In this case the angles $\angle ACB$ and $\angle BCD$ are supplementary and $\angle ACB$ is obtuse. Why? (1 pts.)

Notice that AC and CD are opposite rays and hence $\angle ACB$ and $\angle BCD$ are a linear pair and thus supplementary. As $\angle ACB$ is an exterior angle to $\triangle BCD$ and $\angle CDB$ is a right angle, the exterior angle theorem yields $\angle ACB$ is obtuse.

c. $BD = BC\,(\sin\angle ACB)$ Why? (1 pt.)

Note that, since $\angle BCD$ is acute, $\sin(\angle BCD) = BD/BC$. Now, as $\angle ACB$ and $\angle BCD$ are supplementary, $\sin(\angle ACB) = \sin(\angle BCD)$ and hence $BD = BC\,(\sin\angle ACB)$

d. $CD = -BC\,(\cos\angle ACB)$ Why? (1 pt.)$BD = BC\,(\sin\angle ACB)$

Note that, since $\angle BCD$ is acute, $\cos(\angle BCD) = CD/BC$. Now, as $\angle ACB$ and $\angle BCD$ are supplementary, $\cos(\angle ACB) = -\cos(\angle BCD)$ and hence $CD = -BC\,(\cos\angle ACB)$.

e. $AD = AC + CD$ Why? (1 pt.)

As $A-D-C$, it follows that $AD = AC + CD$

f. $AB^2 = BD^2 + AD^2$ Why? (1 pt.)

Note that $\triangle ABD$ is a right triangle with hypotenuse AB. Thus the Pythagorean theorem yields that $AB^2 = BD^2 + AD^2$.

g. $AB^2 = ((\sin\angle ACB)\,BC)^2 + (AC - (\cos\angle ACB)\,BC)^2$ Why? (1 pt.)

Substitution of $BD = BC\,(\sin\angle ACB)$ and $CD = -BC\,(\cos\angle ACB)$ into (e) and (f).

h. $AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos(\angle C)$ Why? (1 pt.)

$$
\begin{aligned}
AB^2 &= ((\sin\angle ACB)\,BC)^2 + (AC - (\cos\angle ACB)\,BC)^2 \\
&= AC^2 - 2\,(AC)\,(BC)\cos\angle ABC + BC^2\cos^2\angle ABC + B^2C^2\sin^2\angle ABC \\
&= BC^2\left(\cos^2\angle ABC + \sin^2\angle ABC\right) + AC^2 - 2\,(AC)\,(BC)\cos\angle ABC \\
&= BC^2 + AC^2 - 2\,(AC)\,(BC)\cos\angle ABC
\end{aligned}
$$

where the last equality follows from the identity $\sin^2\theta + \cos^2\theta = 1$.

4. Prove one of the following statements: (8 pts.)

a. Using any result prior to Theorem 34: A motion of the form $H_P R_l R_m$ is a quasi-rotation.

b. Using any result prior to Theorem 39: Suppose that l and m are two parallel lines in a Euclidean plane. Prove that there is a line t which is perpendicular to both l and m. (i.e., l and m admit a common perpendicular.)

c. State and then prove the Pythagorean theorem using properties of similar triangles.

These all appear in the notes.

5. Prove one of the following: (12 pts. each)

a. Using any result prior to Theorem 36: Any motion (i.e., composition of finitely many reflections) can be described as the composition of three or fewer reflections.

b. Using the vector definitions for reflections and translations, prove that if l and m are two parallel lines (in $\mathbf{R}^2$), then there is a vector $\overrightarrow{v}$ such that $R_l R_m = T_{\overrightarrow{v}}$.

c. Consider the line $l = \{(x, y) : 2x - 4y = 0\}$ in $\mathbf{R}^2$. Using the taxi-cab distance function, find a coordinate system $f : l \rightarrow R$ for l. (Recall that $d_1((x, y), (w, z)) = |x - w| + |y - z|$.)

The solutions to 5.a. and 5.b. appear in the notes.

For 5.c., the mapping $f\left(x, \frac{1}{2}x\right) = \frac{3}{2}x$ is a coordinate system for l. The steps for showing that it is one-to-one, onto and distance preserving are similar to the solutions that appear in the last progress report.

6. Given the line l, m, and n, sketch the location of P and a such that $R_n R_m R_l = H_P R_a$. (4 pts.)

There are variety of correct solutions. However, for any solution, P must be on the dashed line and the line a must be parallel to the thick line in the diagram given below. (The figure has been distorted in the transfer from GSP to the word processing program. The dashed line and line a should be perpendicular to one another.)

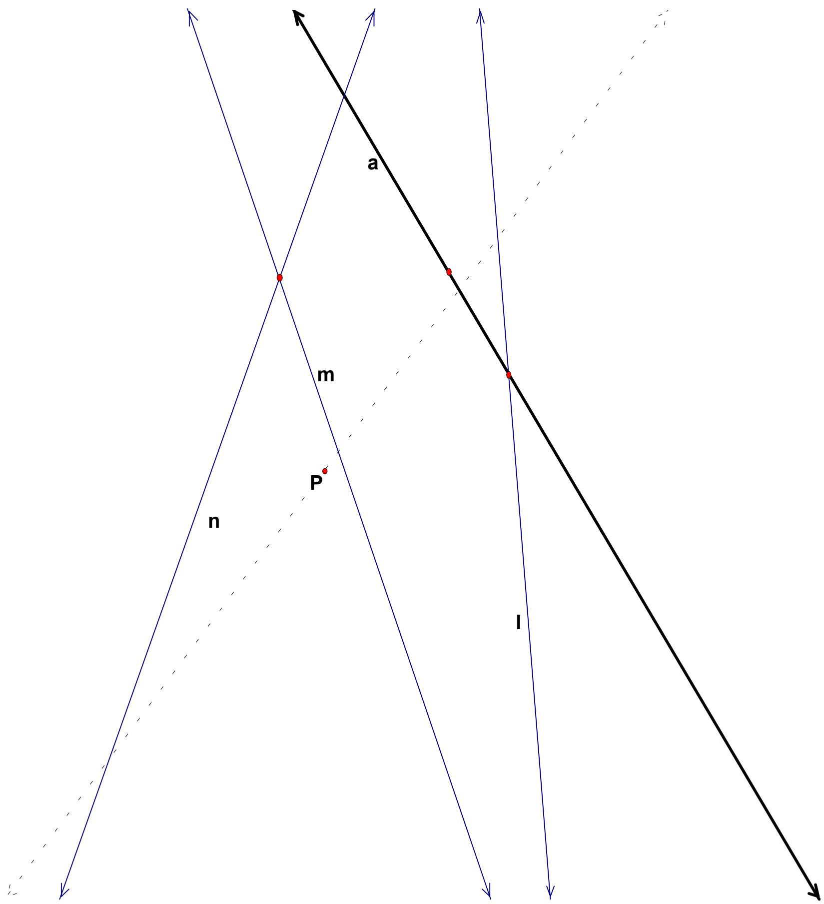

Figure 1: