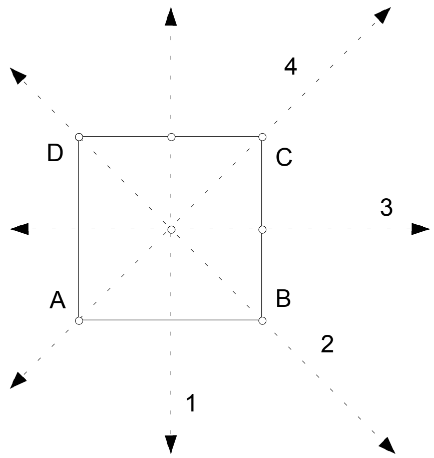


Figure 1:

# Homework 2 - Solutions

1. Using the notation suggested in the following diagram, answer the following questions:

a. Find all of the equivalent descriptions of the mapping $R_1R_2$ and the mapping $R_1R_3$. (4 pts.)

$R_1R_2 = R_2R_3 = R_3R_4 = R_4R_1$. (Notice that these each rotations of 90 degrees in the counter clockwise direction.)
$R_1R_3 = R_2R_4 = R_3R_1 = R_4R_2$. (Notice that these are all half turns.)

b. What is the smallest group containing the reflection $R_1$? Be sure your list only contains one description of each mapping. (4 pts.)

The smallest group (or least of mappings generated by $R_1$) is $\{I, R_1\}$. Here is the multiplication table:

| | $I$ | $R_1$ |
|---|---|---|
| $I$ | $I$ | $R_1$ |
| $R_1$ | $R_1$ | $I$ |

c.. What is the smallest group containing the mappings $R_1$ and $R_2R_4$? Be sure your list only contains one description of each mapping.(4 pts.)

Recall that $R_1R_3 = R_2R_4 = R_3R_1 = R_4R_2$ and hence $R_1(R_2R_4) = R_1(R_1R_3) = R_3$ and that $(R_2R_4)R_1 = (R_3R_1)R_1 = R_3$. It follows that smallest group (or least of mappings generated by $R_1$ and $R_2R_4$ is $\{I, R_1, R_3, R_1R_2\}$. Here is the multiplication table:

| | $I$ | $R_1$ | $R_3$ | $R_2R_4$ |
|---|---|---|---|---|
| $I$ | $I$ | $R_1$ | $R_3$ | $R_2R_4$ |
| $R_1$ | $R_1$ | $I$ | $R_2R_4$ | $R_3$ |
| $R_3$ | $R_3$ | $R_2R_4$ | $I$ | $R_1$ |
| $R_2R_4$ | $R_2R_4$ | $R_3$ | $R_1$ | $I$ |

d. What is the smallest group containing $R_1$ and $R_2$? Be sure your list only contains one description of each mapping.(8 pts.)

This turns out to be all of the symmetries!. The list is

$$I, R_1, R_2, R_3, R_4, R_1R_2, R_2R_1, R_1R_3$$

Here is the multiplication table:

| | $I$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_1R_2$ | $R_2R_1$ | $R_1R_3$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_1R_2$ | $R_2R_1$ | $R_1R_3$ |
| $R_1$ | $R_1$ | $I$ | $R_1R_2$ | $R_1R_3$ | $R_2R_1$ | $R_2$ | $R_4$ | $R_3$ |
| $R_2$ | $R_2$ | $R_2R_1$ | $I$ | $R_1R_2$ | $R_1R_3$ | $R_3$ | $R_1$ | $R_4$ |
| $R_3$ | $R_3$ | $R_1R_3$ | $R_2R_1$ | $I$ | $R_1R_2$ | $R_4$ | $R_2$ | $R_1$ |
| $R_4$ | $R_4$ | $R_1R_2$ | $R_1R_3$ | $R_2R_1$ | $I$ | $R_1$ | $R_3$ | $R_2$ |
| $R_1R_2$ | $R_1R_2$ | $R_4$ | $R_1$ | $R_2$ | $R_3$ | $R_1R_3$ | $I$ | $R_2R_1$ |
| $R_2R_1$ | $R_2R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_1$ | $I$ | $R_1R_3$ | $R_1R_2$ |
| $R_1R_3$ | $R_1R_3$ | $R_3$ | $R_4$ | $R_1$ | $R_2$ | $R_2R_1$ | $R_1R_2$ | $I$ |

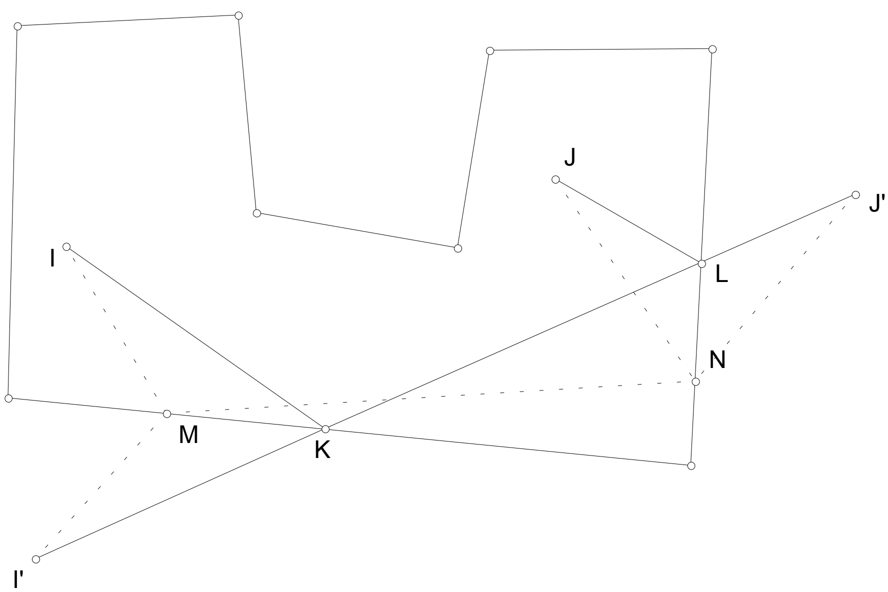


Figure 2:

2. Using your MIRA, carefully plot the path a ball would follow in order to achieve a hole in one. Prove that your path is the shortest path the ball could take to the hole. (6 pts.)

In order to show that $IK + KL + LJ$ is shortest possible (two bank path using the bottom and right hand rails) from $I$ to $J$, one can argue by selecting two points $M$ and $N$ with either $M \neq K$ or $N \neq L$ and then showing that $IK + KL + LJ < IM + MN + NJ$. To argue this, let $I'$ be the reflection of $I$ over the bottom rail and $J'$ be the reflection over right hand rail. Note that $K$ and $L$ are, respectively, the interesections of $\overline{I'J'}$ with the bottom rail and the right hand rail. Now, as reflections are being assumed to preserve distance, two applications of the triangle inequality yield that:

$$\begin{aligned} IK + KL + LJ &= I'K + KL + LJ' \\ &= I'J' \\ &\leq I'M + MJ' \\ &\leq I'M + MN + NJ' \\ &= IM + MN + NJ \end{aligned}$$

As either $M \neq K$ or $N \neq L$, one of the above inequalities must be strict and hence $IK + KL + LJ < IM + MN + NJ$.