Lecture: Distance Functions and the Ruler Postulate

Before moving onto our next topic, it will be useful to discuss the notion of the 'distance between two points'. A geometry with a distance function is called a *metric geometry* and the existence of a distance function allows one to define a variety of basic notions in an intuitively plausible fashion. In addition to discussing abstract properties of distance functions, we will look at how changing the way we compute distance changes which rotations and reflections are isometries. First we will look at the defining properties of a distance function.

Definition: *Let S be a collection of points. A distance function d is a mapping from $S \times S$ into R satisfying the following conditions:*

1. *The mapping d is a function, i.e. each pair of points in $S \times S$ assigned one and only one nonnegative real number.*
2. *For every $A, B \in S$, $d(A,B) = d(B,A)$.*
3. *Given $A, B \in S$, $d(A,B) = 0$ if and only if $A = B$.*

Given a space, there is often more than one way to define the distance between two points in the space. Just for kicks, let us consider a few different ways to define the distance between two points in $\mathbf{R}^2$. First recall

$$\mathbf{R}^2 = \{(x,y) : x, y \in \mathbf{R}\}$$

where $\mathbf{R}$ denotes the set of all real numbers. This is the traditional setting of analytic geometry (of the plane). Recall, from analytic geometry, that lines are sets of the form $\{(x,y) : ax + by + c = 0\}$ where a, b, and c are real numbers, with at least one of a or b being nonzero.

Now for some examples of distance functions. Let $A = (x,y)$ and $B = (z,w)$. Then we set

1. The discrete distance function: $disc(A,B) = 1$ if $A \neq B$ and $d(A,B) = 0$ if $A = B$.
2. The taxi-cab distance function: $d_1(A,B) = |x-z| + |y-w|$ (this is sometimes called the taxi-cab distance or metric)
3. The Euclidean distance function: $d_2(A,B) = \sqrt{(x-z)^2 + (y-w)^2}$
4. The worst-case distance function: $d_\infty(A,B) = \max\{|x-z|, |y-w|\}$

Notice that each of the above rules assigns exactly one nonnegative value to each pair of points and that two points are equal if and only if the distance between them is zero.

Problems 1 - 5 are due Thursday, May 27 at 5:00 p.m. There will be no extensions on this assignment.

Problem 1: For each of the above distance functions, sketch a circle of radius one centered at the origin (i.e., plot the set of points A such that the distance from A to $(0,0)$ is equal to 1.).

Observe that there is no formal connection between lines and the distance function. The 'Ruler Postulate' is used to establish a connection between lines and the distance function and appears, in one form or another, in several secondary school geometry texts. The ruler postulate says, in essence, that any line in the geometry can be placed in a one to one correspondence with the real numbers and that, when restricted to line, the distance function behaves like the absolute value function on the real numbers. This powerful postulate allows us to use the order relations of the real numbers to define betweenness and establish all of the betweenness axioms as theorems in the geometry. It also allows us to define the notion of congruence of segments. Without further ado:

The Ruler Postulate: *Given a geometry satisfying the incidence axioms and having a distance function d. Given any line l there is a one-to-one correspondence* $f : l \to \mathbf{R}$ *such that, for each* $A, B \in l$, $d(A,B) = |f(A) - f(B)|$.

Given a line l and a function f satisfying the conclusion of the ruler postulate, the function f is often called a *coordinate system* for l and, for A on l, the number $f(A)$ is called the *coordinate* of A. Note that distance between points is equal to the distance between the coordinates; because of these, we sometimes say a coordinate system is "distance preserving".

For example, let l be the line $\{(x,y) : y - 3x = 0\}$ in $\mathbf{R}^2$ and suppose that we are using the distance function d_2. Define $f : l \to \mathbf{R}$ by $f(x, 3x) = \sqrt{10}x$. Note that each point is assigned a unique real number, that different points are assigned different real numbers and, given a real number r, there is a point A on l such that $f(A) = r$. [This makes the mapping a one-to-one correspondence]. Now also observer that if $A = (x, 3x)$ and $B = (y, 3y)$, then

$$\begin{aligned} |f(A) - f(B)| &= \sqrt{10}\,|x - y| = \sqrt{10\,(x-y)^2} \\ &= \sqrt{(x-y)^2 + (3x - 3y)^2} = d_2(A,B) \end{aligned}$$

and hence f preserves distance. A line can have more than one coordinate system; for instance, observe that $f(x) = \sqrt{10}\,(x-1)$ is also a coordinate system for l.

The remainder of this project will assume that the geometry has been equipped with a distance function d. In order to save ourselves some work, we let AB denote the distance between A and B. Note that we are starting to develop an elaborate notation system. As you write up your results, keep in mind that $AB, \overline{AB}, \overrightarrow{AB}$ and $\overleftrightarrow{AB}$ all mean different things.

Coordinate Systems and the Ruler Placement theorem

Problem 2: Suppose d_1 is used to find the distance between points in $\mathbf{R}^2$ and $l = \{(x, y) : y - 3x = 0\}$.
a. Find a coordinate system f for l. Given your coordinate system f, which point has coordinate 10?
b. Find a coordinate system g for l such that $g(3, 9) = 0$.

Problem 3: The same as the preceding problem, only using d_∞ as the means of determining the distance between points.

The next problems are of a general nature and require the use of the Ruler Postulate. (Indeed, they are impossible without it.) You may wish draw some inspiration for the solution of problems 4 and 5 from your solutions to problems 2 and 3

Problem 4: Let f be a coordinate system for a line l and define $g : l \to \mathbf{R}$ by $g(A) = -f(A)$ for each $A \in l$. Prove that g is a coordinate system for l.

Problem 5: Prove the *Ruler Placement Theorem*: Let l be a line and A and B two points on l. Then l has a coordinate system in which the coordinate of A is 0 and the coordinate of B is positive.

Problems 6 and 7 are due Thursday, June 3, at 5:00 p.m. Since this is a very short progress report, it is only worth 15 points.

Isometries with respect to different distance functions.

If change the notion of distance, rotations and reflections need not be isometries!

Problem 6: For which angles θ and which lines l through the origin are the motions M_θ (a rotation of θ degrees) and R_l isometries when $\mathbb{R}^2$ is given the taxi-cab distance function? (Use GSP to figure which lines and angles give rise to isometries. Remember to test your conjectures for a wide variety of points.) Find the matrices associated with these rotations and reflections. Explain why, for the lines you found, reflection is an isometry when we use the taxi-cab distance function. (Try to find the underlying cause - this more than reporting your results from GSP.)

Problem 7: This, I think, will be last group process question. Looking back over Math 330A and Math 330B:

- In what way were groups beneficial and in what ways were they not beneficial? How could one modify the way group work is incorporated into the course in order to make it more effective?
- Was it beneficial to develop new material using group work and progress reports? How could the progress reports be restructured so as to be more effective?