

The Classification of Motions: Final Report

It is time to summarize some of your experiences of the last few weeks in a report. In your write-up, present enough detail for a typical member of the class to follow your discussion and use figures to illustrate your arguments every now and then. There is also a section where you are asked to discuss your performance as a cooperative group.

Mathematical Content: The general mathematical theme of this report is the classification of motions of the plane. Recall that a motion is any finite composition of reflections and that, in the last progress report, you showed that any motion - no matter how many reflections are used to define it - can be described using three or fewer reflections. The first results developed along this theme were the theorems on three reflections, which showed that under some circumstances a product of three reflection could be replaced by a single reflection. After developing some properties of half-turns, you showed that certain products of half-turns and reflections could be reduced to a product of two reflections. These results, used in tandem, offer a means of reducing the number of reflections used to describe a motion.

To help with the report, here are some definitions. A motion is called a *proper motion* if it can be described as the product of an even number of reflections, otherwise it is called an *improper motion*. A motion which is the product of two reflections is called a *quasi-rotation*. A *glide-reflection* is a motion of the form $R_n R_m R_l$ where the lines m and l are both perpendicular to n .

In your report, develop the material in the following way:

Prove the following results:

Theorem: *Let P be a point and l be line. Then P is a pole of l if and only if $H_P = R_l$.*

Theorem: *There is a line with a pole if and only if the plane is elliptic.*

In a paragraph, discuss how the first and second theorems on three reflections are used to show that there are a variety of ways of representing translations and rotations, i.e., this can just be a recap of the corollaries following the first theorem and second theorem on three reflections. Also give a complete proof of the following lemma using the Join Theorem.

Lemma: *The motion $H_P R_l R_m$ is a quasi-rotation.*

Now establish:

Theorem: *Every proper motion is a quasi-rotation. Every improper motion is a glide-reflection or a reflection.*

For the first part of the proof, you may wish to review your procedure for showing that an arbitrary motion can be described as the product of three or

fewer reflections. Showing the second part, that an improper motion is either a glide-reflection or a reflection, will take some further work.

Next, we consider the effect of making assumptions regarding parallels. Prove the following:

Theorem: *In an elliptic plane, every motion is a rotation.*

Theorem: *In a Euclidean plane, every proper motion is a rotation or a translation. Every improper motion is a glide reflection or a reflection.*

Conclude your report with a discussion of how, in the Euclidean plane, you can use the fixed points and lines of a motion to help determine if the motion is a reflection, rotation, translation or glide reflection. These statements do not need to be proved, but you should give some discussion as to why you think your claims are correct.

Group Process: What strategies have you developed to determine whether or not the proof of an argument is correct? Illustrate your answer by discussing at least two of the proofs that you did during the last three progress reports. Does working in a group help or hinder the process of determining if a proof is correct?