

## Results related to progress report 7.1

This progress report only had three theorems in it. In order to show that not every point of a fixed line is a fixed point, it is enough to give an example; the easiest place to find an example is probably in the Euclidean plane.

The three theorems can be proven using the axioms up to and including M.2. Note that none of the proofs use any of the parallel postulates.

**Theorem 1** *If  $P$  and  $Q$  are fixed points of the reflection  $R_a$ , then the line determined by  $P$  and  $Q$  is fixed by  $R_a$ .*

**Proof.** Let  $l$  be the line determined by  $P$  and  $Q$ . Since  $R_a$  is a reflection,  $R_a(l)$  is a line. Also note that, as  $P$  and  $Q$  are on  $l$ , the points  $R_a(P)$  and  $R_a(Q)$  are on the line  $R_a(l)$ . Since  $P$  and  $Q$  are fixed points,  $R_a(P) = P$  and  $R_a(Q) = Q$  and hence  $R_a(l)$  contains the points  $P$  and  $Q$ . Since only one line can contain two distinct points,  $R_a(l) = l$ . ■

**Theorem 2** *If lines  $a$  and  $b$  intersect at a point  $P$  and  $a$  and  $b$  are fixed by the reflection  $R_l$ , then  $P$  is a fixed point of  $R_l$ .*

**Proof.** Observe that  $R_l(P)$  is on the lines  $R_l(a)$  and  $R_l(b)$ , and hence the lines  $R_l(a)$  and  $R_l(b)$  meet at  $R_l(P)$ . As  $R_l(a) = a$  and  $R_l(b) = b$ , the lines  $a$  and  $b$  intersect at the point  $R_l(P)$ . As two distinct lines can have at most one point of intersection,  $R_l(P) = P$ . ■

**Theorem 3** *Every line perpendicular to the line  $l$  is fixed by the reflection  $R_l$ . (Note that the line  $l$  is the axis of the reflection.)*

**Proof.** Let  $m$  be a line perpendicular to  $l$  and suppose that  $l$  and  $m$  intersect at a point  $Q$  (Why do we know the point of intersection exists?). Note that, since  $Q$  is on  $l$ , it follows that  $R_l(Q) = Q$ . Also observe that  $R_l(m)$  is a line perpendicular to  $l$  that contains  $Q$ . However, there is only one line perpendicular to  $l$  that contains  $Q$ ; the line  $m$ . Thus  $R_l(m) = m$ . ■

Note that the above proofs make extensive use of the properties of reflections and of perpendiculars. You should make sure that you know which axiom justifies which step.

## The Fixed Line Theorem

The next thing we established was the fixed line theorem and some of its consequences. Recall that a mapping from the plane into itself is called an

*orthogonal collineation* if maps lines into lines and preserves perpendicularity. Reflections, translations and rotations are all examples of orthogonal collineations. Reflections have the properties of an orthogonal collineation by definition and it is straightforward to argue that any composition of a finite number of reflections is an orthogonal collineation.

Before moving on to the substance of this section, first we show that any composition of a finite number of reflections is in fact an orthogonal collineation.

**Proposition 4** *Let  $M$  be a mapping of the plane into itself. If  $M$  can be described as the composition of finitely many reflections, then  $M$  is an orthogonal collineation.*

**Proof.** We prove this by induction. First note that if  $M = R_l$  for some line  $l$  (i.e.,  $M$  is a reflection), then  $M$  is an orthogonal collineation since any reflection maps lines to lines and preserves perpendicularity.

Next we show that if a composition of  $n$  reflections maps lines into lines, then a composition of  $n + 1$  reflections maps lines into lines.

Suppose that  $M = R_{l_{n+1}}R_{l_n}R_{l_{n-1}}\dots R_{l_2}R_{l_1}$  and  $a$  be a line. Note that our assumption yields that  $a' = R_{l_n}R_{l_{n-1}}\dots R_{l_2}R_{l_1}(a)$  is a line and hence, as  $R_{l_{n+1}}$  maps lines into lines,  $R_{l_{n+1}}(a')$  is also a line. Thus

$$\begin{aligned} M(a) &= R_{l_{n+1}}R_{l_n}R_{l_{n-1}}\dots R_{l_2}R_{l_1}(a) \\ &= R_{l_{n+1}}(R_{l_n}R_{l_{n-1}}\dots R_{l_2}R_{l_1}(a)) \\ &= R_{l_{n+1}}(a') \end{aligned}$$

is a line.

The principle of mathematical induction now yields that any finite composition of reflections maps lines into lines.

In order to complete the proof it must be shown that any finite composition of reflections maps perpendicular lines into perpendicular lines. This is left as an exercise. ■

**Lemma 5** *If an orthogonal collineation  $M$  leaves a line  $l$  pointwise fixed and possesses a line  $u$  that is neither equal to  $l$  nor perpendicular to  $l$  as a fixed line, then  $M$  is the identity map.*

**Proof.** The strategy for this proof is to show that every point in the plane can be described as the intersection of two lines that are fixed by  $M$ .

First we show that the line  $u$  given in the hypothesis is *pointwise fixed* by  $M$ . Let  $U$  be a point on  $u$  and drop a perpendicular  $v$  from  $U$  to  $l$ . Let

$Q$  be the point of intersection of  $v$  and  $l$ . Note that  $M(v) = v$ ; this follows from an argument similar to the one given to show that reflections fix lines which are perpendicular to their axis of reflection (given above). Now, since  $U$  is the intersection of lines  $u$  and  $v$  and both of these lines are fixed by the mapping  $M$ , it follows that  $M(U) = U$ ; once again, this follows from an argument similar to the one given in the second theorem in this section. Since  $U$  was an arbitrary point on the line  $u$ , it follows that every point on  $u$  is fixed by  $M$ .

Notice that the above argument shows that for any line  $s$  that is neither perpendicular nor equal to  $l$  and satisfies the property  $M(s) = s$  is pointwise fixed by  $M$ . We will use this observation later in this proof.

Now let  $P$  be any point in the plane; we wish to show that  $M(P) = P$ . Note that if  $P$  is on  $l$ , then the hypothesis yields that  $P$  is a fixed point of  $M$ . To complete the proof, we consider the case that  $P$  is not on  $l$ . Drop a perpendicular  $c$  from  $P$  to  $l$  and let  $C$  denote the point of intersection of  $c$  and  $l$ . Now pick a point  $W$  on  $u$  such that  $W$  is neither on the lines  $c$  nor  $l$ . Let  $d = \overleftrightarrow{WC}$ . Note that  $d$  is not perpendicular to  $l$ , as otherwise  $d = c$ . Also note that, as  $W$  and  $C$  are fixed points of  $M$ ,  $d$  is a fixed line of  $M$ . (Why? Once again, look to the proofs the preceding theorems in this section.) As  $d$  is not perpendicular to  $l$  nor equal to  $l$ , it follows that  $d$  is pointwise fixed. Now drop a perpendicular  $c'$  from  $P$  onto  $d$ . Note that  $c'$  is a fixed line and that  $c \neq c'$  (as otherwise implies that  $d = l$  and thus  $W$  is on  $l$ , a contradiction.). As  $c$  and  $c'$  are both fixed by  $M$  and  $c$  and  $c'$  meet at  $P$ , the point  $P$  is fixed by  $M$ . Since  $P$  was arbitrary,  $M$  is the identity map. ■

**Theorem 6** (*Fixed Line Theorem*) *The only fixed lines of a reflection are the axis of reflection and the lines perpendicular to the axis of reflection.*

**Proof.** Suppose not, i.e. there is a reflection  $R_l$  with axis  $l$  and that  $R_l$  has a fixed line  $u$  that is not equal to  $l$  nor perpendicular to  $l$ . Then  $R_l$  satisfies the hypothesis of the preceding lemma and is hence the identity map. This, however, contradicts the definition of reflection. Thus no such reflection can exist. ■

**Theorem 7** *Let  $l$  and  $m$  be lines. Then  $R_l R_m R_l = R_{m'}$  if and only if  $R_l(m) = m'$ .*

**Proof.** First we show that  $R_l(m) = m'$  implies  $R_l R_m R_l = R_{m'}$ . To do this we show that  $R_l R_m R_l$  satisfies all of the properties of  $R_{m'}$ . First observe

that, since  $R_l R_m R_l$  is a composition of reflections, it maps lines into lines and preserves perpendicularity. To complete the proof we only need to establish that  $m'$  is pointwise fixed by  $R_l R_m R_l$  and that  $R_l R_m R_l$  is not equal to the identity map.

First we show that  $m'$  is pointwise fixed by  $R_l R_m R_l$ . Let  $X'$  be an arbitrary point on  $m'$  and note that  $R_l(X')$  is on  $m$ . This follows from observing that, since  $m' = R_l(m)$ , there must be a point  $X$  on  $m$  such that  $R_l(X) = X'$ . Now note that  $R_l(X') = R_l(R_l(X)) = R_l R_l(X) = X$ , and thus  $X$  is on  $m$ . Next observe that

$$R_l R_m R_l(X') = R_l R_m(X) = R_l(X) = X'$$

and hence  $m'$  is pointwise fixed by  $R_l R_m R_l$ .

Next we show that  $R_l R_m R_l$  is not the identity map. Suppose, for the sake of contradiction, that  $R_l R_m R_l = I$  (where  $I$  denotes the identity map). Then

$$R_m = R_l(R_l R_m R_l)R_l = R_l R_l = I$$

which contradicts the definition of  $R_m$ . Thus  $R_l R_m R_l$  is not the identity map.

As  $R_l R_m R_l$  satisfies all of the properties of  $R_{m'}$ ,  $R_l R_m R_l = R_{m'}$ .

Next we establish the converse. We suppose that  $R_l R_m R_l = R_{m'}$  and then show that  $m' = R_l(m)$ . The first part of this argument shows that if  $u = R_l(m)$ , then  $R_l R_m R_l$  is a reflection with  $u$  as the axis of reflection, i.e.  $R_l R_m R_l = R_u$ . This, combined with the hypothesis  $R_l R_m R_l = R_{m'}$ , yields that  $R_u = R_l R_m R_l = R_{m'}$ . Now a line can be the axis of at most one reflection (see the axioms) and hence  $m' = u = R_l(m)$ . ■

**Theorem 8** *Let  $l$  and  $m$  be two distinct lines. Then  $l$  is perpendicular to  $m$  if and only if  $R_l R_m = R_m R_l$ .*

**Proof.** First observe that the fixed line theorem yields that  $l$  is perpendicular to  $m$  if and only if  $R_l(m) = m$ . Now the preceding theorem yields that  $R_l R_m R_l = R_m$  if and only if  $R_l(m) = m$ . Multiplying both sides on the left by  $R_l$  yields that  $R_l R_m R_l = R_m$  if and only if  $R_l R_m = R_m R_l$ . ■