Results related to the final report

Lemma 1 *Every product* $H_P R_v R_u$ *can be replaced by a product* $R_l R_m$.

Proof. Let $H_P R_v R_u$ be the motion we wish to analyze. We need to consider two possibilities: the lines u and v intersect or the lines u and v are parallel (and may or may not have a common perpendicular.)

First suppose that u and v meet at point Q. Let s be the line containing P and Q and let l be a line perpendicular to s passing through P. Note that $H_P = R_l R_s$ and that, as u, v and s all meet at Q, there is a line m such that $R_m = R_s R_v R_u$. It follows that $H_P R_v R_u = (R_l R_s)(R_v R_u) = R_l (R_s R_v R_u) = R_l R_m$.

Now suppose that u and v do not interesect, and hence create a pencil of the second or third kind. The Join Theorem yields that there is a line s in the pencil formed by u and v that contains the point P. Let l be a line perpendicular to s containing the point P. Note that $H_P = R_l R_s$ and that, as u, v and s are all in the same pencil, there is a line m such that $R_m = R_s R_v R_u$. It follows that $H_P R_v R_u = (R_l R_s)(R_v R_u) = R_l (R_s R_v R_u) = R_l R_m$. ■

Theorem: *Every proper motion is a quasi-rotation. Every improper motion is a glide reflection or a reflection.*

Proof. The proof of the last result of the preceding section shows that every proper motion is a quasi-rotation and that every improper motion can be described as the product of three reflections. To finish the proof, then, all that remains to be shown is that any product of three reflections can be represented as a glide reflection. First recall that any product of three reflections can expressed in the form $H_P R_a$.

Suppose that P is not on a. Now let v be a line perpendicular to a and passing through P and u be a line perpendicular to v and passing through P. Now $H_P R_a = R_v R_u R_a$ and, as u and a are both perpendicular to v, $H_P R_a$ has been described as a glide reflection.

If P is on a, then let b be a line perpendicular to a passing throught P. Then $H_P R_a = R_b R_a R_a = R_b$, which is a reflection. ■

Theorem: *Every motion in an elliptic plane is a rotation.*

Proof. The preceeding theorem yields that every motion is either a quasi-rotation, glide reflection or a reflection. If the motion is a quasi-rotation, then it can be expressed as the product of two reflections $R_l R_m$. Since the plane is elliptic, the lines l and m intersect and hence $R_l R_m$ is a rotation. If the motion is a reflection about a line l, then it can also be describe as a half-turn about the pole of l and hence is a rotation. If the motion is a glide reflection, then it can be described as a reflection followed by a half-turn, i.e., is of the form $H_P R_v$. However the half-turn is also a reflection about the line that has P as a pole, hence $H_P R_v$ can be replaced by the product of two reflections which, as was noted in the beginning of this paragraph, is always a rotation in an elliptic plane. ■

Theorem: *In a Euclidean plane, every proper motion is either a translation or a rotation; every improper motion is a glide reflection or a reflection.*

Proof. Note that we only need to consider the case of proper motions, as the case of improper motions is contained in the more general theorem at the beginning of this section.

Suppose M is a proper motion and hence there are lines l and m such that $M = R_l R_m$. If the lines l and m intersect, then M is a rotation. If l and m do not intersect, then, since we are in the Euclidean plane, they admit a common perpendicular. To see this last claim, pick a point Q on l and let v be a line perpendicular to m containing Q. Now let l' be the line perpendicular to v containing Q. Note that l' cannot intersect m (else the plane would be elliptic) and thus neither l nor l' meet m. Since the plane is Euclidean, $l = l'$, i.e., l is perpendicular to v. ■

Fixed Lines and Fixed Points in the Euclidean plane.

Reflections:

Fixed points: The fixed points are the points on the axis of reflection.

Fixed lines: The axis of reflections and lines perpendicular to the axis.

Rotations which are not half turns:

Fixed points: the center of rotation.

Fixed lines: there are no fixed lines.

Half Turns:

Fixed points:The center of rotation

Fixed lines: Any line passing through the center of rotation.

Translations:

Fixed points: There are no fixed points

Fixed lines: Given translation $R_l R_m$, then any line perpendicular to l is a fixed lines

Glide Reflections:

Fixed points: There are no fixed points

Fixed lines: If l is perpendicular to m and n, then l is fixed by the glide reflection $R_l R_m R_n$