# Math 330B, Spring, 2004: Homework #1

1. Provide support for each line of the proof. Be sure that you have the correct reason. (1 point each.)

**Theorem**: *Let $l$ be a line and $P$ a point not on $l$. Prove that $R_l(P) = P$ if and only if $P$ is a pole of $l$.*

I. Claim: $P$ is a pole of $l$, then $R_l(P) = P$.
a. There are lines $u$ and $v$ that intersect at $P$ which are perpendicular to $l$.
Support: Definition of a pole.

b. The lines $u$ and $v$ are both fixed by $R_l$.
Support: Theorem 3 of the fixed line lecture notes.

c. $R_l(P) = P$.
Support: Theorem 2 of the fixed line lecture notes.

II. Claim: If $P$ is not on $l$ and $R_l(P) = P$, then $P$ is a pole of $l$.

e. Select points $S$ and $T$ on $l$.
Support: Incidence Axiom 2 - a line contains three or more points.

f. The lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{PT}$ exist.
Support: Incidence Axiom 1 - two points determine a unique line.

g. The lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{PT}$ are fixed by $R_l$.
Support: Theorem 1 of the fixed line lecture notes.

h. The lines $\overleftrightarrow{PS}$ and $\overleftrightarrow{PT}$ are both perpendicular to $l$.
Support: Theorem 6 of the fixed line lecture notes. (The fixed line theorem.)

i. $P$ is a pole of $l$.
Support: Definition of a pole.

k. The theorem is proved.
Support: Since we have shown both that if $R_l(P) = P$, then $P$ is a pole of $l$ and if $P$ is a pole of $l$, then $R_l(P) = P$ under the assumption that $P$ is not on $l$, we have shown 'both directions' of the argument.

2. Provide support for each step of the proof of the following theorem. The theorem assumes the betweenness axioms, which appear on the sheet with the axioms for transformational geometry. (1 point each)

**Theorem:** *Suppose the betweenness axioms B.1 - B.6 hold, let $m$ be a line, and $P$ a point not on $m$. If $P'$ is the image of $P$ under the reflection about $m$ and $P \neq P'$, then the line segment with endpoints $P$ and $P'$ intersects $m$.*

a. Drop a perpendicular $v$ from $P$ to $m$.
Support: The second axiom on perpendiculars (P.2).

b. The lines $m$ and $v$ intersect at a point $Q$.
Support: The third axiom on perpendiculars (P.3).

c. The point $P'$ is also on $v$.
Support: Since As $v$ is perpendicular to $m$, we have that $R_m(v) = v$ and hence, as $P$ is on $v$, the point $R_m(P) = P'$ is on $v$. (Theorem 3 of the fixed lines lecture.)

d. Exactly one of the following three betweenness relations must hold:

$$P - Q - P', \qquad Q - P - P', \text{ or } \qquad Q - P' - P.$$

Support: Axiom B.2.

e. If $Q - P - P'$ , then $Q - P' - P$.
Support: Axiom B.6 yields that if $Q - P - P'$, then $R_m(Q) - R_m(P) - R_m(P')$ and, as $R_m(P') = P$ by M.2, we obtain $Q - P' - P$.

f. The assumption $Q - P - P'$ leads to a contradiction.
Support: By B.2, we cannot have $Q - P - P'$ and $Q - P' - P$ being satisfied at the same time.

g. If $Q - P' - P$, then there is a contradiction. (You do not need to support this step.)

h. It follows that $P - Q - P'$.
Support: By B.2, one of the three relations given in step d must be satisfied.

i. The line segment $\overline{PP'}$ intersects $m$.
Support: As $P - Q - P'$, $Q$ is on the line segment $\overline{PP'}$. As $Q$ is on $m$, we now have that $m$ and $\overline{PP'}$ meet at $Q$.

3. Circle the best answer: (1 point each) [These questions have all appeared on previous exams!]

a. If $P$ is a fixed point of a reflection $R_l$ in an elliptical geometry, then $P$ is **sometimes** on the line $l$.

b. If a mapping $M$ is an orthogonal collineation, then there is **sometimes** a point $P$ such that $M(P) = P$.(Consider translations!)

c. It is **false** that if a reflection has a fixed line, then it must be an elliptical geometry.

d. A line in a Euclidean geometry **never** has a pole associated with it.

e. If there is a reflection $R_l$ such that, for any given point $P$, $R_l(P) = P$ implies $P$ is on $l$, then the geometry is **non-elliptic.**

f. If $s, t, l$ and $m$ are distinct lines that meet at a point $P$, then $R_s R_t \neq R_l R_m$ is **sometimes** the case.

g**.** In hyperbolic transformational geometry - If $P$ is a fixed point of a reflection $R_l$, then $P$ is **always** on the line $l$.

h. The composition of two reflections is **sometimes** a reflection.

i. Suppose that $l$ and $m$ are distinct lines on the sphere. If $R_l R_m = R_m R_l$, then $m$ and $l$ are **always** perpendicular.

j. If a line is fixed by a reflection but is not a line of fixed points, then the line is **always** perpendicular to the axis of reflection.

4. Provide a complete proof of the following theorem: (8 pts.)
**Theorem:** *In a nonelliptic plane, a point is a fixed point of a reflection if and only if it is on the axis of reflection.*

Let $l$ be a line and $P$ be a point.

If $P$ is on $l$, then $R_l(P) = P$ by the definition of a reflection with axis $l$.

If $P$ is not on $l$ and $R_l(P) = P$, then $P$ is a pole of $l$ (by question 1 of this homework set) and hence the geometry must be elliptic. This contradicts the assumption that the plane is nonelliptic. Thus if $R_l(P) = P$ then $P$ must be on $l$.

5.Prove the following theorem. (8 pts.)

**Theorem**: *If the betweenness axioms and the plane separation axiom are assumed, then the plane is nonelliptic.*

Proof: We suppose, for the sake of contradiction, that the betweenness axioms hold and that the plane is elliptic.

Let $l$ be a line, $Q$ a point on $l$ and $P$ the pole of $l$. Now let $S$ be a point on $\overleftrightarrow{PQ}$ such that $P - S - Q$. Note that, as it is neither a pole of $l$ nor on $l$, $S$ is not a fixed point of $R_l$. Thus, if we let $S' = R_l(S)$, we have $S' - Q - S$ (by question 2). Since $P$ and $Q$ are fixed points of $R_l$, axiom B.6 yields that $P - S' - Q$. In tandem, $P - S' - Q$ and $S' - Q - S$ yield $P - Q - S$, which contradicts the assumption $P - S - Q$ Thus, if the plane is elliptic, the betweenness axioms cannot hold.