Progress Report: Matrices and Transformations

In this progress report, we will focus on the connection between transformations and matrices. Any linear mapping from $\mathbb{R}^2$ into itself can be represented as 2×2 matrix. It is a nontrivial result that a mapping that preserves distance and maps the origin to itself is a linear map. (In the general case, this result is due to Mazur and Ulam.) The general topic we would like to look at is the connections between matrices and distance preserving maps. A distance preserving map is often called an *isometry* or *rigid motion*.

In class we established that

1. A rotation of θ degrees around the origin can be associated with the matrix
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$
Let M_θ denote a rotation of θ degrees around the origin; what this result says is that if $P = (x, y)$ is a point in $\mathbb{R}^2$ and
$P' = (x\cos\theta - y\sin\theta,\, x\sin\theta + y\cos\theta)$, then $M_\theta(P) = P'$. Keeping the identity $\cos^2\theta + \sin^2\theta = 1$ in mind, we can also think of these matrices as being in the form
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
where $a^2 + b^2 = 1$.

2. If a line through the origin makes an angle of θ degrees with the x-axis, the reflection R_l can be associated with the matrix
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$
Let R_l denote the reflection with axis l; what this result says is that if $P = (x, y)$ is a point in $\mathbb{R}^2$ and
$P' = (x\cos 2\theta + y\sin 2\theta,\, x\sin 2\theta - y\cos 2\theta)$, then $R_l(P) = P'$. Keeping the identity $\cos^2\theta + \sin^2\theta = 1$ in mind, we can also think of these matrices being in the form
$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$
where $a^2 + b^2 = 1$.

One thing we didn't do was establish that rotations and reflections are isometries with respect to the Euclidean distance function. You will do that later in this progress report.

Questions:

1. Using the results above, find the matrices that would be associated with a rotation M_{30} of 30 degrees and a reflection about l,where l makes an angle of 30 degrees with the x-axis. Use the matrices to find the image of $P = (2, 5)$, $Q = (-1, 3)$ and $S = (-2, 3)$ under M_{30} and R_l.

   Solution 1 *This only required 'plugging in' to the formula and applying the resulting matrix to the given points. For instance*

$$
\begin{aligned}
M_{30} &= \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} \\
\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} &= \begin{bmatrix} \sqrt{3} - \frac{5}{2} \\ \frac{5}{2}\sqrt{3} + 1 \end{bmatrix}
\end{aligned}
$$

   and

$$
\begin{aligned}
R_l &= \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \\
\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} &= \begin{bmatrix} \frac{5}{2}\sqrt{3} + 1 \\ \sqrt{3} - \frac{5}{2} \end{bmatrix}.
\end{aligned}
$$

2. Use the reflection and rotation feature of GSP to confirm your answers to question 1.

3. Using the definition of the Euclidean distance function, verify that reflections are distance preserving.

   Proof. Let $P = (x, y)$ and $Q = (w, z)$ and consider the reflection represented by the matrix $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Now let

$$
\begin{aligned}
P' &= \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ -ay + bx \end{bmatrix} \\
Q' &= \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} aw + bz \\ bw - az \end{bmatrix}.
\end{aligned}
$$

Now

$$\begin{aligned}
d(P',Q') &= \sqrt{((ax+by)-(aw+bz))^2+((-ay+bx)-(bw-az))^2} \\
&= \sqrt{(a(x-w)+b(y-z))^2+(-a(y-z)+b(x-w))^2} \\
&= \sqrt{\begin{array}{c}(a^2+b^2)(x-w)^2+(a^2+b^2)(y-z)^2 \\ +2ab(x-w)(y-z)-2ab(x-w)(y-z)\end{array}} \\
&= \sqrt{(x-w)^2+(y-z)^2} = d(P,Q) \qquad (\text{as } a^2+b^2=1)
\end{aligned}$$

■

4. Give a short proof, with no algebra, that the composition of two isometries is an isometry.

   Proof. Let S, T be two isometries with respect to a distance d. We claim that $S \circ T$ is an isometry. Let P and Q be two points and note that

$$\begin{aligned}
d((S\circ T)(P),(S\circ T)(Q)) &= d(S(T(P)),S(T(Q))) \\
&= d(T(P),T(Q)) \\
&= d(P,Q)
\end{aligned}$$

   where the first line follows from the definition of composition and the succeeding lines follow from the definition of an isometry. As P and Q were arbitrary, $S \circ T$ is an isometry. ■

5. Using the results above, show that if l and m are two lines through the origin, then $R_m \circ R_l$ is a rotation. Do this by showing that the product of two matrices of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ ($a^2+b^2=1$) is a matrix of the form $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ ($c^2+d^2=1$).

   Proof. Let $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ and $\begin{bmatrix} c & d \\ d & -c \end{bmatrix}$ be two matrices representing reflections. Now observe that

$$\begin{aligned}
\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} c & d \\ d & -c \end{bmatrix} &= \begin{bmatrix} ac+bd & -(bc-ad) \\ bc-ad & ac+bd \end{bmatrix} \\
&= \begin{bmatrix} e & -f \\ f & e \end{bmatrix}
\end{aligned}$$

where $e = ac + bd$ and $f = bc - ad$. Now observe that

$$\begin{aligned} e^2 + f^2 &= (ac+bd)^2 + (bc-ad)^2 \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= \left(c^2+d^2\right)\left(a^2+b^2\right) = 1 \end{aligned}$$

and hence the product of $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ and $\begin{bmatrix} c & d \\ d & -c \end{bmatrix}$ is a 'rotation matrix'.
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