

Transformations and Matrices in the Cartesian Plane - Lecture Notes

First some definitions and notation. We let $\mathbf{R}^2 = \{(x, y) \mid x, y \in \mathbf{R}\}$ and if $P = (x, y)$ and $Q = (w, z)$ are two points in \mathbf{R}^2 , then

- the (Euclidean) distance from P to Q is $\sqrt{(x - w)^2 + (y - z)^2}$
- The vector \overrightarrow{PQ} is $\langle w - x, z - y \rangle$ with initial point P and terminal point Q . Note that we use angle brackets to denote a vector and round parentheses to denote a point.
- If $P = (x, y)$ and $\vec{v} = \langle v_1, v_2 \rangle$ is a vector, then $P + \vec{v}$ is the point $(x + v_1, y + v_2)$

Now to define the transformations we will be considering.

Definition: A mapping $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is called a translation if there is a vector \vec{v} such that $T(P) = P + \vec{v}$ for every point P in \mathbf{R}^2 . The vector \vec{v} is called the translation vector. The translation with translation vector \vec{v} is sometimes denoted $T_{\vec{v}}$.

Definition: A mapping R is called a reflection if there is a line l such that, for any point P in \mathbf{R}^2 , $R(P) = P + 2\vec{v}_P$ where, if Q is the foot of the perpendicular dropped from P to l , $\vec{v}_P = \overrightarrow{PQ}$. The line l is called the axis of reflection and a reflection with axis l is sometimes denoted R_l .

Note that the translation vector does not depend upon the point being translated. In the definition of reflection, the vector \vec{v}_P does depend upon the point being reflected.

First we establish the connection between translations and reflections.

Theorem: Let l and m be two parallel lines. Then the mapping $R_l \circ R_m$ is a translation.

Proof: Note that we need to find a vector \vec{z} such that $R_l \circ R_m(P) = P + \vec{z}$ for every P in \mathbf{R}^2 . Note that \vec{z} does not depend upon P .

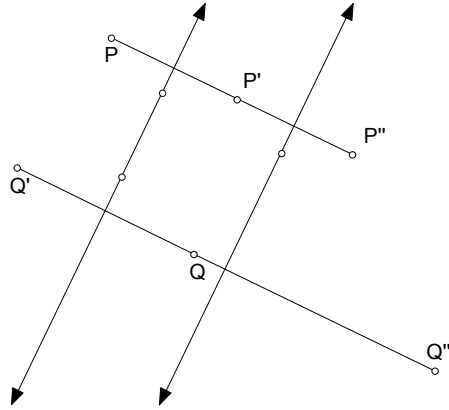


Figure 1:

Suppose that P is given, that $R_m(P) = P'$ where $P' = P + 2\vec{v}_P$, and $R_l(P') = P'' = P' + 2\vec{w}_{P'}$. Then

$$\begin{aligned} R_l \circ R_m(P) &= R_l(R_m(P)) = P' + 2\vec{w}_{P'} \\ &= (P + 2\vec{v}_P) + 2\vec{w}_{P'} \\ &= P + 2(\vec{v}_P + \vec{w}_{P'}) \end{aligned}$$

We claim that the vector $2(\vec{v}_P + \vec{w}_{P'})$ does not depend upon the initial choice of P and hence is the sought after vector \vec{z} .

To see the claim, note that the point $P + \vec{v}_P$ is on the line m and the point $P' + \vec{w}_{P'}$ is on the line l . Note that \vec{v}_P and $\vec{w}_{P'}$ are perpendicular to m and l respectively, and hence parallel. It follows that the vector that would take a point from m to l can be computed as follows:

$$(P' + \vec{w}_{P'}) - (P + \vec{v}_P) = (P + 2\vec{v}_P + \vec{w}_{P'}) - (P + \vec{v}_P) = \vec{v}_P + \vec{w}_{P'}$$

and thus $\vec{v}_P + \vec{w}_{P'}$ is independent of the choice of P .

Observe that $\vec{v}_P + \vec{w}_{P'}$ and $\vec{v}_Q + \vec{w}_{Q'}$ are the same vector in the figure below (hence does not depend on where P or Q was initially chosen.)

Setting $\vec{z} = 2(\vec{v}_P + \vec{w}_{P'})$ now yields that

$$R_l \circ R_m(P) = P + 2(\vec{v}_P + \vec{w}_{P'}) = P + \vec{z} = T_{\vec{z}}(P)$$

for any P in \mathbf{R}^2 and hence $R_l \circ R_m = T_{\vec{z}}$.

We can use the above calculations to establish the converse of the theorem: any translation can be described as the composition of two reflections.

Corollary: *It $T_{\vec{w}}$ is a translation, then there are parallel lines l and m such that $T_{\vec{w}} = R_l \circ R_m$.*

Proof: Let S be any point in \mathbf{R}^2 and let l be the line containing S perpendicular to \vec{w} . Let m be the line

$$l + (1/2) \vec{w} = \{V \mid V = P + (1/2) \vec{w} \text{ for some } P \in l\}$$

Note that the proof of the preceding theorem yields $R_l \circ R_m = T_{2((1/2)\vec{w})} = T_{\vec{w}}$.

Rotations: Rotations centered at the origin have a nice representation in terms of matrices. Let us consider a rotation M centered at the origin that has angle of rotation ϕ . Now let $P = (x, y)$ be any point and suppose that $P = (r \cos \theta, r \sin \theta)$ [think of polar coordinates.] Note that M will rotate (x, y) to the point $(x', y') = (r \cos(\theta + \phi), r \sin(\theta + \phi))$. Hence

$$\begin{aligned} (x', y') &= (r \cos \theta \cos \phi - r \sin \theta \sin \phi, r \sin \theta \cos \phi + r \cos \theta \sin \phi) \\ &= (x \cos \phi - y \sin \phi, x \sin \phi + y \cos \phi) \end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \phi - y \sin \phi \\ x \sin \phi + y \cos \phi \end{bmatrix}$$

Reflections: As in the case of rotations, if a line l passes through the origin there is a nice matrix representation of R_l . Let $P = (x, y) = (r \cos \theta, r \sin \theta)$ be any point and suppose that l makes an angle of ϕ with the x -axis. We would like to get the coordinates of the point $P' = R_l(P)$.

The first difficulty is to get the vector \vec{v}_P to compute $R_l(P) = P + 2\vec{v}_P = P'$. Note that line l has direction $\langle \cos \phi, \sin \phi \rangle$ and hence a vector that will ‘move’ P to l in a direction perpendicular to l has direction $\langle \sin \phi, -\cos \phi \rangle$. [See Figure 2: There is a delicate point here. Be sure to understand why I chose $\langle \sin \phi, -\cos \phi \rangle$ and not $\langle -\sin \phi, \cos \phi \rangle$ (both are perpendicular to $\langle \cos \phi, \sin \phi \rangle$)] Now

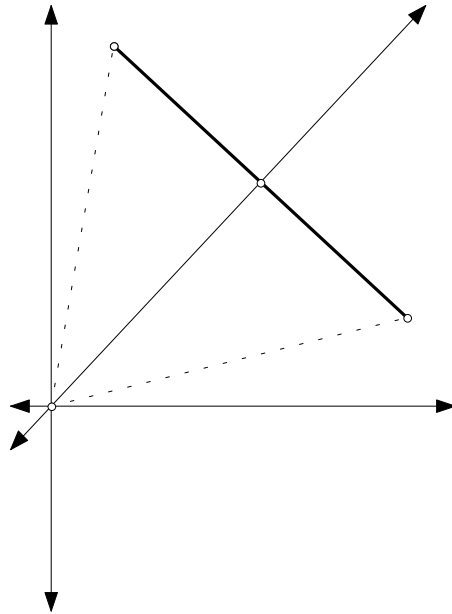


Figure 2:

$$\begin{aligned}
 R_t(P) &= P + 2\vec{v}_P \\
 &= (r \cos \theta, r \sin \theta) + 2r \sin(\theta - \phi) \langle \sin \phi, -\cos \phi \rangle \\
 &= (x', y')
 \end{aligned}$$

Let us do the calculations for x' and y' separately. Before starting, let us recall the trigonometric identities

$$\begin{aligned}
 \sin(\theta - \phi) &= \sin \theta \cos \phi - \sin \phi \cos \theta \\
 \cos(2\phi) &= 2 \cos^2 \phi - 1 = 1 - 2 \sin^2 \phi \\
 \sin(2\phi) &= 2 \sin \phi \cos \phi
 \end{aligned}$$

First we compute x' .

$$\begin{aligned}x' &= r \cos \theta + 2r (\sin \theta \cos \phi - \sin \phi \cos \theta) \sin \phi \\&= r \cos \theta + 2r \sin \theta \cos \phi \sin \phi - 2r \sin \phi \cos \theta \sin \phi \\&= r \cos \theta (1 - 2 \sin^2 \phi) + r \sin \theta (2 \cos \phi \sin \phi) \\&= r \cos \theta \cos (2\phi) + r \sin \theta (\sin (2\phi)) \\&= x \cos (2\phi) + y \sin (2\phi)\end{aligned}$$

and now we compute y'

$$\begin{aligned}y' &= r \sin \theta + 2r (\sin \theta \cos \phi - \sin \phi \cos \theta) (-\cos \phi) \\&= r \sin \theta - 2r \sin \theta \cos^2 \phi + 2r \cos \theta \sin \phi \cos \phi \\&= r \cos \theta (2 \sin \phi \cos \phi) + r \sin \theta (1 - 2 \cos^2 \phi) \\&= x \sin (2\phi) - y \cos (2\phi).\end{aligned}$$

Summarizing the above in matrix form yields

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos 2\phi + y \sin 2\phi \\ x \sin 2\phi - y \cos 2\phi \end{bmatrix}.$$