Results related to progress report 5.2

In this section we make the connection between poles and the elliptic parallel postulate.

Theorem 1 *If P is a pole of l and m is perpendicular to l, then P is on m.*

Proof. Suppose that u and v are both perpendicular to l and that they meet at P. Since u and v are fixed by R_l, we have that P is a fixed point of R_l. Now suppose that m meets l at a point Q. Since P and Q are both fixed points of R_l, it follows from the results of the preceding section that $\overleftrightarrow{PQ}$ is a fixed line of R_l and thus, by the fixed line theorem, $\overleftrightarrow{PQ}$ is perpendicular to l (as P is not on l). Since $\overleftrightarrow{PQ}$ and m are both perpendicular to l and both meet l at Q, the uniqueness of perpendiculars yields that $\overleftrightarrow{PQ} = m$ and thus P is on m. ■

Theorem 2 *If P is a pole of l and P is on a line m, then l is perpendicular to m.*

Proof. Select a point Q on m and drop a perpendicular m' from Q onto l. The previous result shows that P is on m'. As m and m' both contain P and Q, $m = m'$ and hence m is perpendicular to l. ■

Theorem 3 *A line has at most one pole.*

Proof. Suppose P and Q are poles of a line l. Suppose that u and v are two lines perpendicular to l that meet at P. The first result of this section shows that Q is on both u and v. As u and v can meet in at most one point, $P = Q$. ■

Theorem 4 *If one line has a pole, then every line has a pole.*

Proof. Suppose that l is a line with a pole P and that m is a line. We wish to show that m has a pole. There are two cases to consider: m is perpendicular to l and m is not perpendicular to l.

First we consider the case m is perpendicular to l. Note that P is on m and construct a line c perpendicular to m passing through P. Since P is on

c, it follows that c is perpendicular to l and hence c and l must intersect; let S denote this point of intersection. The point S is the pole of m.

Note that the above argument shows that any line perpendicular to a line with a pole also has a pole.

Next suppose that m is not perpendicular to l. In this case P is not on m. Drop a perpendicular d from P to m; observe that since P is on d, the line d is perpendicular to l. Now construct a line e that contains P and is perpendicular to d. Let T denote the point of intersection of e and l and observe that T is the pole of the line d. Now, since m is perpendicular to d and d has a pole, m has a pole. ■

Theorem 5 *Given lines l and m with poles P and Q. The l and m intersect at the pole of the line $\overleftrightarrow{PQ}$.*

Proof. First note that since one line has a pole, then every line has a pole and hence $\overleftrightarrow{PQ}$ has a pole. Now note that l and m are both perpendicular to $\overleftrightarrow{PQ}$ (as P is on $\overleftrightarrow{PQ}$ and Q is on $\overleftrightarrow{PQ}$). Thus l and m meet at the pole of $\overleftrightarrow{PQ}$. ■

Lecture: Theorems on Three Reflections

Theorem 6 *(First theorem on three reflections). If the lines l, m, and n intersect at a point P, then there is a line w passing through P such that $R_n R_m R_l = R_w$.*

Proof. First note that if $m = n$, then $R_n R_m R_l = R_l$ and the conclusion of the theorem follows. Thus we may assume, without loss of generality, that $m \neq n$. Now note that axiom M.3 guarantees that there is a line d such that $R_n R_m (P) = R_d (P)$ for every point P on l and that m, n and d are concurrent Note that this implies that $R_d R_n R_m (P) = P$ for every point P on l. As $R_d R_n R_m$ is an orthogonal collineation, it follows that either $R_d R_n R_m = R_l$ or $R_d R_n R_m = I$.

Observe that if $R_d R_n R_m = R_l$, then it follows immediately that $R_n R_m R_l = R_d$ and the conclusion of the theorem is satisfied.

To complete the proof, then, it suffices to show that $R_d R_n R_m = I$ leads to a contradiction. If $R_d R_n R_m = I$, then $R_d (R_d R_n R_m) = R_n R_m = R_d \Longrightarrow$

$R_nR_d = R_m$ and $R_d = (R_dR_nR_m)(R_mR_n) = R_mR_n \Rightarrow R_dR_n = R_m$ and hence $R_m = R_nR_d = R_dR_n$. Thus d is perpendicular to n (note that if $n = d$, then $R_m = I$, a contradiction). A similar calculation can be used to show that m is perpendicular d. As there is only one perpendicular to d at the point where m, n, and d intersect, it follows that $m = n$. This, however, contradicts the assumption that $m \neq n$ and hence $R_dR_nR_m \neq I$. ■

Theorem 7 *If $M = R_lR_m$ is a rotation about P, then given any line u passing through P there is another line v passing through P such that $M = R_vR_u$.*

Proof. Let l, m, and u be as in the statement of the theorem. The first theorem on three reflections yields that there is a line v passing through P such that $R_lR_mR_u = R_v$. It is immediate that $R_lR_m = R_vR_u$. ■

Theorem 8 *(Second theorem on three reflections). If the lines l, m, and n are all perpendicular to a line t, then there is a line z perpendicular to t such that $R_nR_mR_l = R_z$.*

Proof. This proof is very similar to Theorem 1.

First note that if $m = n$, then $R_nR_mR_l = R_l$ and the conclusion of the theorem follows. Thus we may assume, without loss of generality, that $m \neq n$. Now note that axiom M.4 guarantees that there is a line d such that $R_nR_m(P).= R_d(P)$ for every point P on l and that m, n and d are perpendicular t. Note that this implies that $R_dR_nR_m(P) = P$ for every point P on l. As $R_dR_nR_m$ is an orthogonal collineation, it follows that either $R_dR_nR_m = R_l$ or $R_dR_nR_m = I$.

Observe that if $R_dR_nR_m = R_l$, then it follow immediately that $R_nR_mR_l = R_d$ and the conclusion of the theorem is satisfied.

To complete the proof, then, it suffices to show that $R_dR_nR_m = I$ leads to a contradiction. If $R_dR_nR_m = I$, then $R_d(R_dR_nR_m) = R_nR_m = R_d \Longrightarrow R_nR_d = R_m$ and $R_d = (R_dR_nR_m)(R_mR_n) = R_mR_n \Rightarrow R_dR_n = R_m$ and hence $R_m = R_nR_d = R_dR_n$. Thus d is perpendicular to n (note that if $n = d$, then $R_m = I$, a contradiction). A similar calculation can be used to show that m is perpendicular d. Note that m, n,and d intersect at the pole of t. As there is only one perpendicular to d at the point where m, n, and d intersect, it follows that $m = n$. This, however, contradicts the assumption that $m \neq n$ and hence $R_dR_nR_m \neq I$. ■

Note that it is not enough to show that m is perpendicular to d to arrive at a contradiction; it could well be that m and d meet at the pole of t.

Theorem 9 *If $T = R_l R_m$ is a translation along t and u is a line perpendicular to t, then there is a line v perpendicular to t such that $T = R_v R_u$.*