Results related to Half-Turns and Pencils of Lines

Definition 1: *A half turn about a point P is a rotation R_mR_l where the axes of the reflections l and m are perpendicular and intersect at P.*

Theorem 2: *If H_P is a half-turn, then $H_P = R_vR_u$ for any pair of perpendicular lines u and v that meet at P.*

Proof. Suppose that $H_P = R_lR_m$ where l and m are a given pair of perpendicular lines meeting at P. Recall that $R_lR_m = R_mR_l$. Now, we have from the preceding section that there is a line v' passing through P such that $R_lR_m = R_{v'}R_u$. Observe that

$$R_mR_lR_lR_m = R_mR_lR_{v'}R_u = I \Longrightarrow R_lR_m = R_mR_l = R_uR_{v'}$$

and hence $R_{v'}R_u = R_uR_{v'}$. It follows that $u \perp v'$ and hence, as there can be only one perpendicular to u at P, $v = v'$. Thus $H_P = R_vR_u$. ■

Theorem 3: *Points and half-turns are in a one-to-one correspondence.*

Theorem 4: *Every line passing through P is a fixed line under H_P.*

Proof. Suppose that s is a line passing through P. Let t be a line perpendicular to t passing through P. Note that, as $s \perp t$, $R_t(s) = s$. It follows that $H_P(s) = R_sR_t(s) = R_s(s) = s$. As s was arbitrary line passing through P, H_P fixes every line passing through P. ■

Theorem 5: *$H_P = R_l$ if and only if P is a pole of l.*

Proof. First suppose that P is a pole of l and let Z be a point on l. Let $u = \overleftrightarrow{PZ}$ and let v be line perpendicular to u passing through P. Note that v is perpendicular to l and hence Z is a pole of v. It follows that $R_v(Z) = Z$. Now $H_P(Z) = R_uR_v(Z) = R_u(Z) = Z$. As Z was an arbitrary point of l, l is pointwise fixed by H_P.

Also note that H_P is an orthogonal collineation not equal to the identity map, thus $H_P = R_l$.

Now suppose that $H_P = R_l$ and let u and v be two perpendicular lines meeting at P. Note that u and v are both fixed by H_P and hence by R_l. It follows that $u \perp l$ or $u = l$. If $u = l$, it follows that R_v is the identity map (a contradiction), hence we must have that $u \perp l$. A similar argument shows that $v \perp l$. As u and v are distinct perpendiculars that meet at P it follows that P is the pole of l. ■

The proof of the next theorem is omitted as it is fairly straightforward.

Theorem 6: *The only fixed lines of* H_P *are*

1. *all lines passing through the point* P *together with the polar line of* P *if the plane is elliptic.*
2. *all the lines passing through* P *if the plane is nonelliptic.*

Theorem 7: *The only fixed points of* H_P *are:*

1. P *and all the points of the polar line of* P *if the plane is elliptic.*
2. P *alone if the plane is not elliptic.*

Proof. If the plane is elliptic, the previous result yields that $H_P = R_l$ where P is the polar line of l. As the only fixed points of R_l are the points on l and the pole of l $(=P)$, the result follows.

Next we observe that if there is a point P such that there is a point Q with the properties that $P \neq Q$ and $H_P(Q) = Q$, then there is a line with a pole and hence the geometry is elliptic. (We are making no assumptions about parallel lines at this point in the argument.) Let $u = \overleftrightarrow{PQ}$ and v be a line perpendicular to u which intersects v at P. Note that, $Q = H_P(Q) = R_u R_v(Q) = R_u(Q)$ - the last equality comes from noting that Q on v implies $R_v(Q) = Q$ - and hence Q is either a pole of u or on u. If Q were on u,then we would have $u = v$, which contradicts the assumption that u and v are perpendicular. Hence Q is the pole of u and, as there is a line with a pole, the plane is elliptic.

Thus, if the plane is non-elliptic, the only fixed point of H_P is P. ■

Theorem 8: *Let* P *be a point and* l *be a line. Then* $P' = R_l(P)$ *if and only if* $H_{P'} = R_l H_P R_l$.

Proof. Suppose that $H_{P'} = R_l H_P R_l$. We wish to show that $P' = R_l(P)$. Suppose that u and v are pair of perpendicular lines that meet at P and let u' and v' denote $R_l(u)$ and $R_l(v)$ respectively. Observe that $u' \perp v'$ and hence meet a point Q. Also note that the construction yields that $R_l(P) = Q$. We will show that $H_Q = H_{P'}$ and hence $P' = Q$. First recall, from project 5, that $R_{u'} = R_l R_u R_l$. Now:

$$H_Q = R_{u'} R_{v'} = (R_l R_u R_l)(R_l R_v R_l) = R_l R_u R_v R_l = R_l H_P R_l = H_{P'},$$

where the last equality is the hypothesis of this direction of the proof. Thus $P' = Q$.

Next we assume that $P' = R_l(P)$ and show that $H_{P'} = R_l H_P R_l$. Let u be a line perpendicular to l passing through the point P and let v be a line perpendicular to u passing through P. Note that $R_l(u) = u$ and that, if v' denotes $R_l(v)$, then $u \perp v'$. Observe that P' is at the intersection of u and v'. Now note that

$$H_{P'} = R_u R_{v'} = R_u (R_l R_v R_l) = R_l R_u R_v R_l \quad \text{(since } u \perp v\text{)} = R_l H_P R_l.$$

(Note that this version of the proof does not require two separate cases (one for P on l and one for P not on l). ■

Definition 9: *Let M be a motion. The motion M is called an involution or said to be involutoric if $M \neq I$ and $M^2 = I$.*

The following theorem is an immediate consequence of the preceding theorem.

Theorem 10: *Let P be a point and l be a line. Then P is on l if and only if $H_P R_l$ is involutoric.*

Proof. First suppose that $H_P R_l$ is involutoric. Then $I = (H_P R_l)(H_P R_l)$ and hence $H_P = R_l H_P R_l$ (as $H_P H_P = I$). Thus it follows from the preceding theorem that $R_l(P) = P$ and hence either P is on l or P is the pole of l. But if P is the pole of l, then $H_P = R_l$ and thus $H_P R_l = I$, which contradicts the definition of involutoric. Consequently, P must be on l.

Now suppose that P is on l. Then $R_l(P) = P$ and hence $H_P = R_l H_P R_l$. It follows that $I = (H_P R_l)(H_P R_l)$. As P is not a pole of l, $H_P \neq R_l$ and hence $H_P R_l \neq I$. The $H_P R_l$ is involutoric. ■

Theorem 11: *Let l and m be distinct lines belonging to a pencil of lines through a point P. A line x belongs to the pencil of lines through P if and only if $R_x R_m R_l$ is a reflection.*

Proof. That P on x implies that $R_x R_m R_l$ is a reflection was established in class (First Theorem on Three Reflections.).

We show that if $R_x R_m R_l$ is a reflection, then P is on x. Suppose that P is not on x and drop a perpendicular v from P to x; let P' denote the intersection of x and v. Since v, m, and l meet at P, there is a line q such that $R_q = R_v R_m R_l$ and P is on q. Note that $R_v R_q = R_m R_l$ and that

$$R_x R_m R_l = R_x R_v R_v R_m R_l = H_{P'} R_q.$$

As $R_x R_m R_l$ is a reflection, $H_{P'} R_q$ is involutoric and thus P' is on q. Now note that P and P' are on both q and v and hence $q = v$. However, since $R_v R_q = R_m R_l$, this forces $l = m$. which is a contradiction. Hence P is on x. ■

Definition 12: *Let l, m be lines. Then the set* $\{x : x \text{ is a line and } R_x R_m R_l \text{ is a reflection}\}$ *is the pencil of lines determined by l and m.*

A pencil of the first kind *is a set of lines that all have one point in common.*

A pencil of the second *kind is the set of all lines perpendicular to a given line.*

A pencil of the third kind *is a pencil which is not of the first or second kind.*

Theorem 13:*Let l and m belong to a pencil of lines perpendicular to a given line t. A line x belongs to the pencil of lines perpendicular to t if and only if $R_x R_m R_l$ is a reflection.*

Proof. This is similar to the proof of the preceding theorem - just use the corresponding results for lines with a common perpendicular. ■

The proof of the following theorem is a good bit of work, hence we omit it.

Theorem 14: *(Join Theorem) A pencil of the first kind and a pencil of the second or third kind have one and only one line in common.*