Parallel projections and the basic similarity theorem

In this lecture we develop some of the preliminary material necessary to discuss the properties of similar triangles. First, of course, a definition.

Definition 1 *Let ΔABC and ΔDEF be two triangles. The correspondence $ABC \leftrightarrow DEF$ is a similarity provided that corresponding angles are congruent and*

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

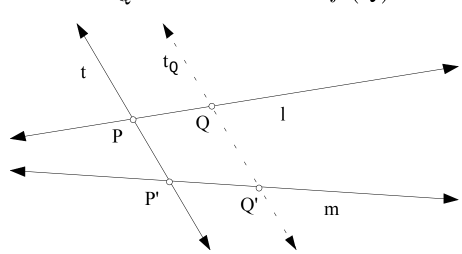
In the event the correspondence $ABC \leftrightarrow DEF$ is a similarity, we write $\Delta ABC \sim \Delta DEF$ and say that the triangles ΔABC and ΔDEF are similar.

The results in this lecture and developed in this series of progress reports are essentially Euclidean in nature. We will be adopting all of the axioms of absolute geometry and the Euclidean parallel postulate throughout (except in progress report 5.2, where we call the area axioms into play again). Let us recall some of the consequences of the Euclidean parallel postulate. These include:

1. In any plane, two lines parallel to a third line are parallel to each other.
2. Given two lines and a transversal. If the lines are parallel, then each pair of corresponding angles is congruent.
3. The sum of the measures of a triangle is equal to 180.
4. In a parallelogram, each pair of opposite sides are congruent to one another.

These will each play a role in the discussion to come. First we introduce the definition of a parallel projection and see how even this definition relies on the Euclidean parallel postulate.

Let l and m be two lines with a transversal t. (Note that these three lines are coplanar by the definition of transversal.) We define *the parallel projection f of l onto m in the direction of t* as follows: If P is the intersection of l and t and P' is the intersection of m and t, we define $f(P) = P'$. Now suppose the Q is a point on l, $P \neq Q$. Let t_Q be a line parallel to t passing through Q and let Q' be the intersection of t_Q and m. Define $f(Q)$ to be the point Q'.



That Q' is well-defined relies upon the parallel postulate. Note that we need to show that t_Q and m intersect in a single point. We do this in two steps:

1. the lines t_Q and m intersect: Suppose not. Then $t \| t_Q$ and $t_Q \| m$, and hence $t \| m$. This, however contradicts the definition of transversal.

2. the lines t_Q and m are not equal: If $t_Q = m$, then $t \| m$, which once again contradicts the definition of transversal.

Theorem 2 *A parallel projection is a one-to-one correspondence.*

Theorem 3 *Parallel projections preserve betweenness.*

Theorem 4 *Parallels projections preserve congruence.*

Theorem 5 *(The Basic Similarity Theorem) Let l, m and n be three parallel lines with transversals t and t' intersecting them at points A, B and C and A', B' and C' respectively. If $A - B - C$, then*

$$\frac{BC}{AB} = \frac{B'C'}{A'C'}.$$

Theorem 6 *If two segments on the same line have no point in common, then the ratio of their lengths is the same under every parallel projection.*

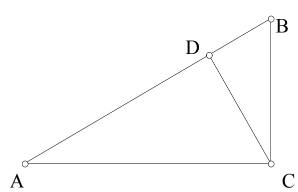
Theorem 7 *Parallel projections preserve ratios.*

The Pythagorean theorem

Definition 8 *An altitude of a triangle is a perpendicular segment from a vertex to the line containing the opposite side.*

Note that every triangle has three altitudes (draw a sketch to convince yourself.) Also recall that if ΔABC is a right triangle with right angle at C, then $\overline{AB}$ is the hypotenuse of the triangle and, if D is the foot of the perpendicular dropped from C to $\overleftrightarrow{AB}$, then $A - D - B$. (This is a consequence of the theorem which asserts that in a triangle the greater side is opposite the greater angle.)

The following diagram reflects the notation for the next two proofs.



Theorem 9 *The altitude to the hypotenuse of a right triangle divides the triangle into two triangles, each of which is similar to the right triangle.*

Proof. Let ΔABC be a right triangle with right angle at C and let D be the foot of the perpendicular dropped from C to the line containing $\overline{AB}$. Recall that $A-D-B$, hence the altitude $\overline{CD}$ divides the triangle ΔABC into two triangles ΔACD and ΔCBD. Now the AA criteria for the similarity of triangles (Euc) yields that

$$\Delta ABC \sim \Delta ACD \sim \Delta CBD.$$

■

Theorem 10 *(Pythagorean Theorem) In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.*

Proof. Using the notation from the preceding proof, we have that

$$\frac{AD}{AC} = \frac{AC}{AB} \text{ and } \frac{BD}{BC} = \frac{BC}{AB}$$

and hence

$$AD = \frac{1}{AB}(AC)^2 \text{ and } BD = \frac{1}{AB}(BC)^2.$$

It follows that

$$AB = AD + BD = \frac{1}{AB}(AC)^2 + \frac{1}{AB}(BC)^2$$

and hence $AB^2 = AC^2 + BC^2$. ■

Theorem 11 *Given a triangle which has sided of lengths a, b and c. If $a^2+b^2 = c^2$, then the triangle is a right triangle with right angle opposite the side of length c.*

Proof. Let ΔABC be the given triangle and suppose that a is the length of the side opposite A, b is the length of the side opposite B, and c is the length of the side opposite C. Construct a right triangle ΔDEF with right angle at F such that $EF = a$ and $DF = b$. The Pythagorean theorem yields that $DE^2 = EF^2 + DF^2 = a^2 + b^2 = c^2$, hence $DE = c$. Now by the SSS criteria for the congruence of triangles, $\Delta ABC \cong \Delta DEF$ and hence ΔABC is a right triangle with right angle at C. As c is opposite C, we are done. ■

Theorem 12 *Given two right triangles. If the hypotenuse and one leg of one triangle are congruent to the hypotenuse and one leg of the other, then the triangles are congruent.*

Proof. This follows from the Pythagorean theorem and the SSS criteria for the congruence of triangles. ■

Theorem 13 *In any triangle the product of a base and the corresponding altitude is independent of the choice of altitude and base.*

Proof. Let ΔABC be a triangle, $\overline{BD}$ an altitude from B to $\overleftrightarrow{AC}$, and $\overline{CE}$ an altitude from C to $\overleftrightarrow{AB}$. We need to show that $(CE)(AB) = (BD)(AC)$ or, equivalently, $\frac{AB}{AC} = \frac{BD}{CE}$. Note that,by AAA similarity, $\Delta AEC \sim \Delta ADB$ and hence $\frac{AB}{DB} = \frac{AC}{AE}$. ■

Theorem 14 *For similar triangles, the ratio of any two corresponding altitudes is equal to the ration of any two corresponding sides.*

Proof. Similar to the preceding arguments. ■