Axioms and Definitions of Transformational Geometry

Incidence Axioms:

Primitive terms: point, line

I.1 Given two different points, the points lie on one and only one line.

I.2 Each line contains at least three distinct points.

I.3 There are at least three noncollinear points.

Perpendiculars:

Primitive terms: perpendicular

P.1 Let l and m be lines. If l is perpendicular to m, then m is perpendicular to l.

P.2 If P is any point and l is any line, then there is a line m containing P such that l is perpendicular to m. If the point P is on l, then m is uniquely determined.

P.3 If l is perpendicular to m, then l and m intersect in a single point.

Motions:

Definition: Let l be a line. A reflection R_l (in l) is a one-to-one mapping of the plane onto itself with the following properties:

a. Lines are mapped onto lines.

b. If two lines are perpendicular, then their reflected images are perpendicular.

c. Every point of l is mapped onto itself.

d. There is a point of the plane which is not mapped onto itself.

For a point P in the plane, we let $R_l(P)$ denote the image P under R_l. The line l is called the axis of R_l.

M.1 Every line is the axis of one and only one reflection.

M.2 If R_l maps the point P to the point P', then R_l maps P' onto P.

Definition: A mapping M from the plane to itself is called a *rotation* if there are two intersecting lines u and v such that $M = R_u R_v$ (i.e., M is the composition of two reflections). The point where u and v intersect is called a *center* of the rotation.

M.3 Consider a rotation M with center P. If l is a line containing P, then there is a reflection R_m, with P on m, that $R_m(Q) = M(Q)$ for every Q on l.

Definition: A mapping M from the plane to itself is called a *translation* if there are two nonintersecting lines u and v both perpendicular to a line t such that $M = R_u R_v$ (i.e., M is the composition of two reflections R_u and R_v) In this case M is called a translation along t.

M.4 Consider a translation M along a line t. If l is a line perpendicular to t, then there is a reflection R_m with m perpendicular to t, such that $R_m(P) = M(P)$ for every P on l.

Parallels

Euclid Given a line l and a point P not on l, there is a unique line containing P that does not intersect l.

Elliptic Any two lines intersect.

Hyperbolic Given a line l and a point P not on l, there are at least two different lines containing P that do not intersect l.

Betweenness and Separation

Primitive terms: between. $P - Q - R$ denotes Q is between P and R

B.1 Let P, Q, and R be points. If $P - Q - R$, then $R - Q - P$.

B.2 Given three distinct collinear points, exactly one is between the other two.

B.3 Any four points can be named in an order P, Q, R, and S in such a way that $P - Q - R - S$.

B.4 If P and R are any two points, then a) there is a point X such that $P - R - X$ and b) there is a point Y such that $P - Y - R$.

B.5 If P, Q, and R are points and $P - Q - R$, then P, Q, and R are collinear.

B.6 If R_m is a reflection and P, Q, and S are points such that $P - Q - S$, then $R_m(P) - R_m(Q) - R_m(S)$

Plane Separation Given a line and a plane containing it, the set of all points in the plane can be written as the disjoint union of the line and two convex sets with the property that if P belongs to one of the convex sets and Q belongs to the other, then the line segment with endpoints P and Q intersects the line.