Homework #3 - Solutions

Problem 1: Using the SAS criteria for similarity (see the first page), prove the 'midsegment theorem': Let ΔABC be a triangle and suppose that D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{BC}$. Prove that $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{AC}$ and $DE = \frac{1}{2}AC$.

Solution: First note that since D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{BC}$, we have that

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{1}{2}.$$

Since $\angle B$ is congruent to itself, the SAS criteria for similarity yields that $\Delta ABC \tilde{} \Delta DBF$. Now $\angle BAC$ is congruent to $\angle BDE$ and hence (as $\overleftrightarrow{BA}$ is transversal to $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$ and forms a pair of corresponding congruent angles) $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{AC}$. The similarity also yields that $DE/AC = 1/2$ and hence $DE = \frac{1}{2}AC$.

Problem 2 :Prove the *Law of Sines:* If ΔABC is any triangle, then

$$\frac{AB}{AC} = \frac{\sin(\angle C)}{\sin(\angle B)}$$

In order to prove this, let D be the foot of the perpendicular dropped from A to $\overleftrightarrow{CB}$. The key to the proof is to use compute AD using $\sin(\angle C)$ and AC, compute AD using $\sin(\angle B)$ and AB, set the results equal to one another, and then simplify. Notice that there are a number of cases that need to be considered: $B-C-D$, $C-D-B$, $D=B$, $D=C$ and $D-B-C$.

a. Prove the result when $C-D-B$. Include a diagram that illustrates this case. (4 pts.)

First note that, in this case, $\angle C$ and $\angle B$ are acute angles. To see this, note that since $C-D-B$ we have that $\overrightarrow{CD} = \overrightarrow{CB}$ and hence $\angle ACD = \angle ACB$. As ΔACD is a right triangle with right angle at D, it follows that $\angle ACD$ and hence $\angle ACB$ is acute. A similar argument shows that $\angle ABC$ is acute.

Now

$$AC \sin\angle ACB = AD = AB \sin\angle ABC$$

and the result follows.

b. Prove the result when $C-B-D$. Include a diagram that illustrates this case.(4 pts.)

In this case $\angle C$ is acute and $\angle B$ is obtuse. To see this note that $C-B-D$ yields that $\overrightarrow{CB} = \overrightarrow{CD}$ and hence $\angle ACB = \angle ACD$. As ΔACD is a right triangle with right angle at D, it follows that $\angle ACD$ and hence $\angle ACB$ is acute. Now note that $C - B - D$ yields that $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are opposite rays; hence $\angle ABD$ and $\angle ABC$ are a linear pair and hence supplementary. As ΔACD is a right triangle with right angle at D, it follows that $\angle ABD$ is acute and hence $\angle ABC$ is obtuse.

Now

$$AC \sin \angle ACB = AD = AB \sin \angle ABD$$

and, as $\sin \angle ABD = \sin \angle ABC$ by definition,

$$AC \sin \angle ACB = AD = AB \sin \angle ABC$$

and the result follows.