ON D-SPACES

TODD EISWORTH

ABSTRACT. We take a look at current research surrounding van Douwen's D-spaces.

INTRODUCTION

So what makes a mathematical problem interesting? This question always provokes spirited debate among mathematicians no matter where it is posed. Of course there is no good answer to it (as English critic and author William Hazlitt says, "Whatever interests, is interesting."), but there are certainly some mathematical questions that arouse the curiosity of almost anyone who comes in contact with them, questions that tempt with the simplicity of of their formulation, tantalize with promises of an elegant solution if only one can look at the problem in just the right way, and taunt with the number of excellent mathematicians who have examined the question in the past and failed to solve it. The theory of D-spaces is replete with such questions, and in this short note we will examine a few of them.

What *is* a D-space? A topological space[1] X is a *D-space* if for every *neighborhood assignment* $\{N(x) : x \in X\}$ (that is, $N(x)$ is an open neighborhood of x for each $x \in X$) there is a closed discrete subset D of X such that $X = \bigcup\{N(x) : x \in D\}$. The concept goes back to work of van Douwen [15]. One of the first things said about D-spaces in the cited paper is the following:

> Up to now no satisfactory example of a space which is not a D-space is known, where by satisfactory example we mean an example having a covering property at least as strong as metacompactness or subparacompactness.

Over twenty years later, the situation is much the same. We still lack a basic understanding of the relationship between covering properties and the state of being a D-space. In fact, as Fleissner and Stanley noted in 2001 [12]:

> Besides the trivial observation that a compact T_1-space is a D-space, there are no proofs known that a covering property implies D-space.

This state of affairs is the main topic of the following short note.

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[1]We assume that all spaces under consideration are at least regular.

Questions about D-spaces

The first subject we address is our lack of understanding about the relationship between covering properties and the state of being a D-space. As Fleissner and Stanley say, we simply lack theorems that say that such-and-such covering property implies that a space is a D-space. Moreover, we lack the techniques that would allow us to build counterexamples to such alleged theorems. To wit, all of the following problems (essentially from [15]) remain open:

? 1001 **Question.** *Is every (hereditarily) Lindelöf space a D-space?*

? 1002 **Question.** *Is every paracompact space a D-space?*

? 1003 **Question.** *Does there exist a subparacompact or metacompact space which is not a D-space?*

In view of the last question, we recall that a space X is *subparacompact* if every open covering of X can be refined by a σ-discrete closed covering.

Arhangelskii [3] has recently addressed the relationship between covering properties and D-spaces. He adds the following questions to van Douwen's list:

? 1004 **Question** (Problem 1.18 of [3])**.** *Is every countably metacompact weakly θ-refinable (Tychonoff) space a D-space?*

Recall that a space X is weakly θ-refinable if for each open covering $\mathcal{C}$ of X, there exists a sequence $\langle \mathcal{C}_n : n < \omega \rangle$ of open coverings of X, each refining $\mathcal{C}$, such that for every $x \in X$ there is a $k < \omega$ with $\mathcal{C}_k$ point-finite at x.

A σ-metrizable space is weakly θ-refinable, so this suggests the related question:

? 1005 **Question** (Problem 1.21 of [3])**.** *Is every countably metacompact σ-metrizable space a D-space?*

Finally, let us recall that a space X is *screenable* if every open covering of X has an open σ-disjoint refinement. Arhangelskii and Buzyakova [1] establish that every space with a point countable base is in fact a D-space; in particular, every space with a σ-disjoint base is a D-space, and so the following question is natural:

? 1006 **Question** (Problem 1.22 of [3])**.** *Is every screenable (Tychonoff) space a D-space?*

We next turn to a problem of Buzyakova concerning cardinal invariants and their relation to D-spaces. We start with the observations that every compact space is trivially a D-space, and that a countably compact D-space is compact. These facts follow immediately from the easy fact that $l(X) = e(X)$ for a D-space (where $l(X)$, the *Lindelöf number of* X, is the smallest infinite cardinal τ such that every open covering of X contains a subcovering of cardinality $\leq \tau$, and $e(X)$, the *extent of* X, is defined to be the supremum of cardinalities of closed discrete subsets of X).

The converse is not true; Buzyakova notes in [9] that if we take the product $X = D(\omega_1) \times \omega_1$ (where $D(\omega_1)$ is a discrete space of cardinality ω_1), then $l(X) = e(X) = \aleph_1$, but X is not a D-space. To see this last fact, note that X contains a closed copy of ω_1 and it is straightforward to see that this precludes X being a D-space. However, she notes that the following question may be of interest:

Question (Question 3.6 of [9]). *Suppose that $l(Y) = e(Y)$ for every subspace Y of X. Is X then a D-space?* **? 1007**

It is worth mentioning Buzyakova's main result from [9]: if X is compact, then $C_p(X)$ (the space of continuous real-valued functions on X with the topology of pointwise convergence) is hereditarily a D-space. Her result is quite strong; for example, it allows one to immediately deduce the following two well-known theorems:

Theorem (Baturov [4]). *If X is compact, then $l(Y) = e(Y)$ for every subspace Y of $C_p(X)$.*

Theorem (Grothendieck [13]). *If X is compact and Y is a countably compact subspace of $C_p(X)$, then Y is compact.*

Of course, both theorems follow immediately from Buzyakova's result using simple properties of D-spaces, and perhaps this helps to make the case that D-spaces are a class worthy of more research.

Arhangelskii also puts forward the following question regarding D-spaces and spaces of the form $C_p(X)$:

Question (Question 1.23 of [3]). *Suppose that X is a Tychonoff space with $C_p(X)$ Lindelöf. Is $C_p(X)$ then a D-space?* **1008 ?**

Many other questions on D-spaces have recently appeared in the literature and we do not have space to consider them all. We refer the reader instead to [3], [2], [1], and [9] for more comprehensive coverage, and limit ourselves to the following intriguing questions of Arhangelskii concerning unions of D-spaces.

Question. *Suppose that a (regular, Hausdorff, Tychonoff) T_1-space is the union of two subspaces which are both D-spaces. Is then X a D-space as well?* **1009 ?**

Question. *Suppose X is countably compact, and $X = \bigcup_{n<\omega} X_n$ where each X_n is a D-space. Is X compact?* **1010 ?**

Arhangelskii conjectures that a positive answer to the first is highly unlikely, but he and Buzyakova [1] have shown that if a regular T_1 space X is the union of a finite collection of metrizable subspaces, then X is a D-space. Regarding the second question, Gary Gruenhage [14] has shown that a positive answer results if we require that X is a finite union of D-spaces.

Stickyness

The fact that Gruenhage's [14] ends the previous section is serendipitous, for we want to examine his techniques in more detail in this section. In [14], Gruenhage develops a general framework based on earlier work of Fleissner and Stanley [12] and implicit in Buzyakova's work that seems to handle the most important theorems of the form "a space X of such-and-such a type must be a D-space". He uses these methods to solve many open problems asked by Arhangelskii, Buzyakova, and others. We briefly outline his techniques, as they should be helpful to anyone tackling problems in this area.

Let X be a space. A binary relation R on X is *nearly good* if $x \in \overline{A}$ implies that $x \, R \, y$ for some $y \in A$. If N is a neighborhood assignment on X, and X' and D are subsets of X, then we say that *D is N-sticky mod R on X'* if whenever x is in X' and $x \, R \, y$ for some $y \in D$, then in fact $x \in N(D)$, i.e., $N(D)$ swallows all R-predecessors of each $y \in D$ that lie in X'. If $X' = X$, then we say[2] that *D is N-sticky mod R*.

The simplest example illustrating this definition is to let N be a neighborhood assignment on X, and define

$$(*) \qquad x \, R \, y \Longleftrightarrow y \in N(x).$$

This particular R is nearly good, and a set D is N-sticky mod R if $x \in N(D)$ whenever $N(x) \cap D \neq \emptyset$.[3]

Theorem (Gruenhage)**.** *Let N be a neighborhood assignment for X. Suppose as well that R is a nearly good relation on X such that every non-empty closed subset F of X contains a non-empty closed discrete subset D that is N-sticky mod R on F. Then there is a closed discrete D^* in X with $X = N(D^*)$.*

This theorem has some strength. Consider, for example, the case where X is left-separated and N is a neighborhood assignment on X, without loss of generality with $N(x) \subseteq [x, \infty)$ (the interval is defined using the order that left-separates X). Let R be as in $(*)$. Given a non-empty closed subset F of X, let x be the least element of F. Then the closed discrete set $D = \{x\}$ is N-sticky mod R on F, and so from the preceding theorem we conclude that X is a D-space.

We get more powerful results using the next theorem, which is also taken from [14]. The statement of the following theorem makes reference to N-close sets, where N is a neighborhood assignment on X. We say that a subset Z of X is *N-close* if $Z \subseteq N(x)$ for every $x \in Z$.

Theorem (Gruenhage)**.** *Let N be a neighborhood assignment on X, and suppose there is a nearly good relation R on X such that for any $y \in X$, we can express the set $R^{-1}(y) \setminus N(y)$ as a countable union of N-close sets. Then there is a closed discrete D such that $X = N(D)$.*

We give one more easy example from [14] illustrating how powerful this result is. The key is that one can vary the relation R in order to get results in different situations.

Recall that a space X satisfies *open* (G) if each point $x \in X$ has a countable neighborhood base $\mathcal{B}_x$ such that whenever $x \in \overline{A}$ and $N(x)$ is a neighborhood of x, then there is an $a \in A$ and $B \in \mathcal{B}_a$ for which $x \in B \subseteq N(x)$.

[2]Gruenhage deals with a generalization of this situation, where R is a relation from X to $[X]^{<\omega}$; this generalization allows him to capture more examples, but we shall deal only with the simpler version.

[3]This is what Fleissner and Stanley referred to as "N-sticky" in [12].

Theorem. *Any space satisfying open* (G) *is a* D*-space.*

Proof. Given a neighborhood assignment N for such a space X, one defines

$$x \, R \, y \iff \text{there exists a } B \in \mathcal{B}_y \text{ such that } x \in B \subseteq N(x).$$

This choice of R is nearly good because X satisfies open (G). Furthermore, for each $B \in \mathcal{B}_y$, we can let $C(B)$ be the set of all $x \in B$ for which $B \subseteq N(x)$. The set $C(B)$ is N-close, and $R^{-1}(y) = \bigcup\{C(B) : B \in \mathcal{B}_y\}$. From the theorem cited earlier, we conclude that X is a D-space. □

It is clear that any space with a point-countable base satisfies open (G)[4] and so such spaces are D-spaces, a fact first shown by Arhangelskii and Buzyakova [1].

Gruenhage's paper contains a wealth of other results; for example, he shows that all Corson compacta are hereditarily D-spaces, and that $C_p(X)$ is hereditarily a D-space whenever X is a Lindelöf Σ-space. We refer the reader to [14] for the details.

General remarks

The current state of knowledge about D-spaces is full of asymmetries. We are rich with theorems that state that certain types of spaces are D-spaces, but we are lacking theorems of the form "If X is a D-space, then . . . ". We have many results that state that spaces with certain types of bases are D-spaces, but there are no substantial theorems saying that spaces satisfying certain covering properties are D-spaces. We have fairly general techniques for proving that something is a D-space, but we are sorely in need of more techniques for building spaces that are not D-spaces. Correcting these asymmetries should provide the next generation of general topologists with ample work.

Finally, I thought I would drop the authorial "we" for a moment, just to say that I, too, pulled out a pencil and scrap paper when I first heard the question of whether a regular Lindelöf space must be a D-space. I was sure I could see how the proof would go, and then later that night I reversed my opinion and thought I could see how a counterexample might work. Good problems are like this—they are interesting because they interest!

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