A.1 Introduction (for non-mathematicians)

A dialogue

When I discuss my research with non-mathematicians, the conversation will usually run along the following lines (with me in the role of questioner):[1]

Q: "The best way to understand what it is I do is to think about the following: What is the most fundamental concept of mathematics?"

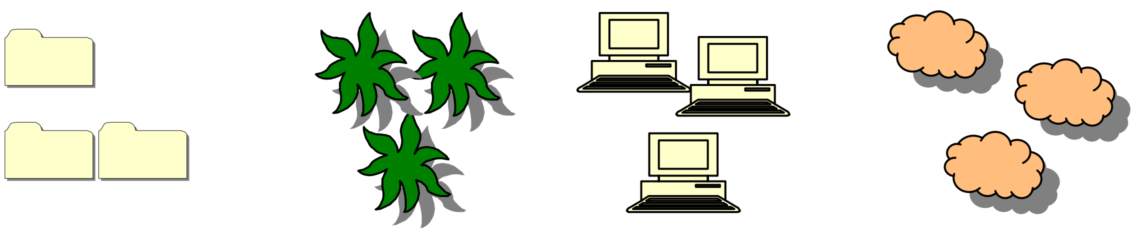
A: "Numbers."

Q: "Is this the most basic, most fundamental idea in mathematics, or is there something else underlying even the concept of 'number'?"

A: "Not that I can see."

Q: "How does a child learn the identity of the number 'three'? How does one first grasp 'three-ness'?"

A: "Usually by presenting them with several groups of three objects, counting each group individually, and hoping for a leap of understanding."



Q: "So you might present a child with something like the pictures above?"

A: "Yes. The groups of three objects let me demonstrate 'three-ness'. 1-2-3 1-2-3 1-2-3"

Q: "You're now at the heart of the matter – you used the idea of a 'group of objects'."

A: "What do you mean?"

Q: "Before you even began to count, you mentally grouped the like objects together, and 'three-ness' is an abstract property possessed by each of the groups."

A: "So?"

Q: "The mental act of considering those objects as organized in 'groups' is necessary before one can even start to get across the concept of 'three'."

A: "So you claim that this 'mental operation' is a more fundamental concept than even something like the number three? So what? What does this have to do with mathematics?"

Q: "It has everything to do with mathematics. It can be used as the *very foundation* of mathematics as we know it."

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[1] Any resemblance to a well-known conversation about geometry between a famous Greek philosopher and an Athenian slave boy is purely coincidental.

Sets and the foundations of mathematics – a cartoon sketch

> *By a "set" we mean any collection M into a whole of definite, distinct objects m (which are called the "elements" of M) of our perception or of our thought.*
>
> \- Georg Cantor

The imaginary dialogue of the last section tried to explain how something like the ability to mentally form "collections of objects" (or "sets", as we shall henceforth refer to them) can claim to be a very rudimentary manifestation of mathematics. Georg Cantor (1845-1918) was the first mathematician to attempt a mathematical treatment of sets; his work greatly enlarged the mathematical universe and revealed that many of the concepts we associate with finite collections (like "size" or "counting") can be extended to the realm of the infinite. For our purposes, however, it is the work of Gottlob Frege (1848-1925) that needs attention. Frege realized that by taking Cantor's concept of set in combination with the predicate logic that he (Frege) was developing, one could derive an account of arithmetic: the theory of sets, together with logic, seemed to serve as an adequate foundation for mathematics.

I say "seemed", because in 1903 Bertrand Russell discovered what came to be known as "Russell's Paradox" – a relatively simple (but diabolically clever) argument that shows Cantor's notion of "set" is incompatible with basic logic. The vague idea of a "set" as a "collection of objects" is demonstrably incoherent provided we accept rudimentary logic as valid.

Because Frege's treatment of arithmetic[2] in terms of sets seemed so appealing, the first decades of the Twentieth Century were full of attempts to rescue his project from the grasp of Russell's Paradox (e.g., Russell and Whitehead's *Principia Mathematica*). There were several approaches adopted; the one that concerns us is the attempt to confront the vagueness of Cantor's concept of set through axiomatization. The hope was that Russell's Paradox could be circumvented by carefully specifying exactly what sorts of Cantor's 'vague collections' should be admitted into mathematics. The particular collection of axioms used by most set theorists today is called ZFC – "Zermelo-Fraenkel with Choice". One should think of these axioms as a very precisely stated set of rules specifying the properties that mathematical sets must possess. The axioms are rich enough to guarantee that Frege's project can be carried out while apparently avoiding Russell's Paradox. With the isolation of these axioms, mathematics again appeared to be firmly situated inside "Cantor's Paradise". Once again, appearances were deceiving.

Gödel and incompleteness – a caricature

> *Aus dem Paradies, das Cantor uns geschaffen hat, soll uns niemand vertreiben können*.
> (No one can expel us from the Paradise that Cantor has created for us).
>
> -- David Hilbert

I will not even try to state Gödel's two famous Incompleteness Theorems precisely; I will caricature them by saying simply that axiom systems such as ZFC (and again, I won't make precise exactly what properties the axiom systems must satisfy) are doomed to underdetermine the mathematical objects they are trying to capture.

It is easier to understand this if we take a Platonist vision of mathematics and pretend that there is a metaphysical realm in which all the abstract objects of mathematics and geometry enjoy a real existence. In this garden of delights, we will find things like π, fractals, and isosceles triangles. It is here that we will find an object corresponding to (or responsible for, if we really buy into the Platonism) our idea of "three-ness" – we will find Three itself.

Further exploration of this realm uncovers the existence of objects we might call "true sets" – the objects behind our mathematical idea of sets. The collection of true sets is quite special to us, for it is rich enough to contain copies of almost all the other objects in this Platonic wonderland – this is an

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[2] And hence (using work of Cauchy, Dedekind, and others) most of mathematics as then conceived.

outgrowth of Frege's work discussed earlier. Let us then distinguish this special collection by dubbing it the "universe of true sets".

Back in the mundane world, the axioms of ZFC can be viewed as an attempt to capture the essence of the universe of true sets – the intent is that these axioms write down everything we feel entitled to say about the mathematical sets of our intuition; the hope is that we have such a firm grasp on how sets behave that our universe of true sets will be the only thing in the Platonic realm satisfying all the demands of our axioms. Gödel's work tells us our hope is in vain – any potential axiom system that is simple enough for us to understand necessarily cannot bear the weight our hopes would ask of it.

The nature of my research

This (at last) brings me to the nature of my research. For the sake of argument, let us agree that ZFC represents mankind's best effort at plumbing the nature of sets. Gödel's Incompleteness Theorem implies that there are necessarily many "faux universes" scattered throughout the Platonic landscape – collections of objects that satisfy all our intuitive ideas about sets without actually being the universe of true sets. Said another way, we have no way of deciding whether any one of these things is actually the universe of true sets or not – from our point of view, ZFC codifies everything we think we understand about the true sets, and these faux universes all satisfy the axioms. Moreover, within each of these universes we can develop mathematics based on these faux sets – all we need to know is that the objects we are working with satisfy our axioms. Does this matter? Do we get different versions of mathematics depending on which of these faux universes we take as home? This is disconcerting, given the propensity of mathematics to lay claim to "Truth" with a capital T!

Things are not nearly as dramatic as my metaphysical fable makes them out to be. It will always be the case that 1 + 1 = 2 no matter which "mathematical universe" happens to be the true one – although the actual identity of the objects we are calling "1" and "2" might be different from universe to universe, the relationships between the objects are invariant. This is the beauty of axiomatics – as long as the axioms are satisfied, all of the mathematical consequences of the axioms will hold. Thus, if we equate "ordinary mathematics" with "things that can be proved from the axioms of ZFC", then it follows that ordinary mathematics is something common to all of our candidates for the universe of true sets.

However, it is not the case that all mathematics is ordinary. In almost every major field of pure mathematics, there are instances where natural lines of investigation have led to questions whose resolution lies beyond the ken of "ordinary mathematics" – questions whose answer can change from universe to universe. Such questions are the subject of my research. In particular, my work centers on the task of taking specific questions (from set theory, or other areas of mathematics) and demonstrating mathematically that one can neither prove nor refute them by using only "ordinary mathematics".

We come now to the end of this introductory section. In a sense, everything I have written above is false – I have oversimplified some things and over-dramatized others in an attempt to explain to a non-mathematician what it is I do and why I find it interesting. The next section will offer a much more technical (and mathematically precise) discussion of the specifics of my work.