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MATH 263B: Exercises on the Fundamental Theorem of Calculus

In order to derive maximum benefit from this handout, you should do each exercise before reading the subsequent text.

In class you learned about the following theorem:

Theorem 1 (Fundamental Theorem of Calculus, Part I) *If f is a continuous function on an interval $[a, b]$, then the function F defined for x such that $a \leq x \leq b$ by*

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

is continuous on $[a, b]$ and differentiable on (a, b), with

$$\frac{dF}{dx}(x) = f(x). \tag{2}$$

Exercise 2 *Let $f(t) = t^2$, and let $F(x)$ be the area of the region bounded by the t-axis, the graph of the function f, and the vertical lines $t = 1$ and $t = x$. Find $\frac{dF}{dx}(5)$.*

Now let us consider the function $F(x)$ defined in Exercise 2. This function is defined as:

$$F(x) = \int_1^x t^2\,dt. \tag{3}$$

The Fundamental Theorem of Calculus tells us that $\frac{dF}{dx}(x) = x^2$, and thus the function $F(x)$ is an antiderivative of the function $f(x) = x^2$. From our discussion of antiderivatives it now follows that $F(x) = \frac{x^3}{3} + C$ for some constant C. In order to figure out what C is, notice that $F(1) = \int_1^1 t^2\,dt = 0$. It follows that

$$F(1) = 0 = \frac{1^3}{3} + C. \tag{4}$$

Solving Equation (4) for C we find that $C = -\frac{1}{3}$, and we conclude that $F(x) = \frac{x^3}{3} - \frac{1}{3}$.

Exercise 3 *Let $f(t) = e^t$, and let $F(x)$ be the area of the region bounded by the t-axis, the graph of the function f, and the vertical lines $t = 1$ and $t = x$. Find the formula for $F(x)$.*

Exercise 4 *Let $f(t) = 2^{-t}$, and let $F(x)$ be the area of the region bounded by the t-axis, the graph of the function f, and the vertical lines $t = -2$ and $t = x$. Find the formula for $F(x)$.*

In the formulation of Exercises (2)–(4) we have not given much thought about the domain of the function $F(x)$. In particular, Formula (3) makes a whole lot of sense if $x \geq 1$; but does it still hold if $x < 1$? A moment's thought reveals that this is not the case. If $x < 1$ then the area of the region bounded by the t-axis, the graph of the function $f(t) = t^2$, and the vertical lines $t = 1$ and $t = x$ is $\int_x^1 t^2\,dt$, not $\int_1^x t^2\,dt$. Thus the formula $F(x) = \frac{x^3}{3} - \frac{1}{3}$ will be valid only for x from the interval $[1, \infty)$, not for all possible x's.

Exercise 5 *For which intervals are your solutions of Exercises 3 and 4 valid?*

Now let us return to the function $F(x)$ defined in Exercise (2), and let us try to find a formula for it on the interval $(-\infty, 1)$. As mentioned above, on this interval we have $F(x) = \int_x^1 t^2\,dt$. The problem is that here the x is the *lower limit of integration,* while the Fundamental Theorem of Calculus only tells us about the situation where x is the *upper limit of integration.* Now here is a challenge for you: Formulate and prove a modification of the Fundamental Theorem of Calculus that deals with this situation! Here is a template for this theorem:

Theorem 6 (Modified Fundamental Theorem of Calculus, Part I) *If f is a continuous function on an interval $[a, b]$, then the function F defined for x such that $a \leq x \leq b$ by*

$$F(x) = \int_x^b f(t)\,dt \tag{5}$$

is continuous on $[a, b]$ and differentiable on (a, b), with

$$\frac{dF}{dx}(x) = ???. \tag{6}$$

Exercise 7 *Replace the question marks in Theorem 6 by a formula so that you get a valid statement. Then prove the resulting theorem.* Hint: *You may use Theorem 1 in your proof, as well as the fact that $\int_b^a f(t)\,dt = -\int_a^b f(f)\,dt$.*

Exercise 8 *Use the result of Exercise 7 to derive formulas for the functions $F(x)$ of Exercises 2 for the arguments x in the intervals $(-\infty, 1)$. Recall that we already established that $F(x) = \int_x^1 t^2\, dt$ for these arguments x.*

Exercise 9 *use the result of Exercise 7 to derive formulas for the functions $F(x)$ of Exercises 3 and 4 for the arguments x in the intervals $(-\infty, 1)$ and $(-\infty, -2)$ respectively.*