MATH 263B, Winter Quarter 2005

Integration by parts

Earlier in this course, we discussed the "sum rule for integrals" and you may have wondered whether there is such a thing as a product rule for integrals. Such a rule does indeed exist; it goes by the name of "integration by parts."

Consider a product of two functions $f(x)$ and $g(x)$, that is, consider the function $f(x)g(x)$. Then the product rule for derivatives states that $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$. Taking antiderivatives of both sides we get:

$f(x)g(x) = \int (f(x)g(x))'\,dx = \int f'(x)g(x) + f(x)g'(x)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$

By rearranging the terms the latter formula can also be written as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This formula is called *the formula for integration by parts.* It is perhaps easier to remember in the following notation. Let u and v be two new variables such that $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$, and the formula for integration by parts can be written as follows:

$$\int u\,dv = uv - \int v\,du.$$

The formula for integration by parts may not seem particularly useful at first glance. After all, the formula only allows us to express one integral in term of another integral; it does not give an explicit recipe for solving any integral. But as we will show in a moment, the formula is very useful indeed for computing a number of important integrals.

Example 1: $\int 2xe^x\,dx$.

The integrand in this example can be written as a product $f(x)g'(x)$, where $f(x) = 2x$ and $g'(x) = e^x$. Then $f'(x) = 2$ and $g(x) = e^x$, and the formula for integration by parts gives:

$$\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + C.$$

Why did we choose $f(x) = 2x$ and $g'(x) = e^x$ rather than $f(x) = e^x$ and $g'(x) = 2x$? The answer is that the latter choice is perfectly legal, but it does not help us in evaluating the integral. We would have $f'(x) = e^x$ and $g(x) = x^2$, and the formula for integration by parts would give the the following:

$$\int 2xe^x\,dx = x^2e^x - \int x^2e^x\,dx.$$

This is not wrong, but it is useless, since the integral on the right hand side is even more complicated than the integral on the left hand side. When you try to solve an integral by integration by parts and the initial choice of $f(x)$ and $g'(x)$ does not appear to lead anywhere, it may be a good idea to return to the beginning and try reversing the roles of f and g.

Example 2: $\int x \sin x \, dx$.

Let us write the integrand as $f(x)g'(x)$ where $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. Now the formula for integration by parts gives:

$$\int x \sin x \, dx = -x \cos x - \int 1(-\cos x) \, dx = -x \cos x + \sin x + C.$$

Homework 1: Evaluate the following integrals:

(a) $\int xe^{-x} \, dx$.

(b) $\int x \cos x \, dx$.

(c) $\int x \ln x \, dx$.

(d) $\int 2x^3 e^{x^2} \, dx$ *Hint:* Let $g'(x) = 2xe^{x^2}$.

(e) $\int x^{-\frac{1}{3}} e^{x^{\frac{1}{3}}} \, dx$ *Hint:* Let $g'(x) = \frac{1}{3}x^{-\frac{2}{3}} e^{x^{\frac{1}{3}}}$.

Sometimes it is necessary to apply the formula for integration by parts several times in a row.

Example 3: $\int x^2 e^{2x} \, dx$.

Let $f(x) = x^2$ and $g'(x) = e^{2x}$. Then $f'(x) = 2x$ and $g(x) = 0.5e^{2x}$. The formula for integration by parts yields:

$$\int x^2 e^{2x} \, dx = 0.5x^2 e^{2x} - \int 2x(0.5)e^{2x} \, dx = x^2 e^x - \int xe^{2x} \, dx.$$

The integrand on the right hand side can be written as $f_1(x)g_1'(x)$, where $f_1(x) = x$ and $g_1'(x) = e^{2x}$. Then $f_1'(x) = 1$ and $g_1(x) = 0.5e^{2x}$, and integration by parts gives:

$$\int xe^{2x} \, dx = 0.5xe^{2x} - \int 0.5e^{2x} \, dx = 0.5xe^{2x} - 0.25e^{2x} + C.$$

Now the original integral evaluates to:

$$\int x^2 e^{2x} \, dx = 0.5x^2 e^{2x} - 0.5xe^{2x} + 0.25e^{2x} + C.$$

Example 4: $\int e^x \sin x \, dx$.

Let $f(x) = \sin x$ and $g'(x) = e^x$. Then $f'(x) = \cos x$ and $g(x) = e^x$. Applying integration by parts we get:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

The integrand on he right hand side can be written as $g_1'(x) f_1(x)$, where $g_1'(x) = e^x$ and $f_1(x) = \cos x$. Then $g_1(x) = e^x$ and $f_1'(x) = -\sin x$. Integration by parts gives:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx.$$

Now the exact same integral pops up on the left hand side and on the right hand side of the equation and it may appear that we have not accomplished anything. But wait: What if we add $\int e^x \sin x \, dx$ to both sides of the above equation? This gives:

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C,$$

and we conclude that

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

The sudden appearance of the constant C in the above reasoning may seem a bit mysterious, but one should realize that when the symbol $\int e^x \sin x \, dx$ is written on both sides of an equation, then it may represent *different* antiderivatives of the function $e^x \sin x$. This possibility is being accounted for by adding C to the right hand side when the integral is moved to the left hand side.

Homework 2: Evaluate the following integrals:

(a) $\int x^2 \sin x \, dx$.

(b) $\int x^3 e^x \, dx$.

(c) $\int x^5 e^{x^2} \, dx$.

(d) $\int x^3 (\ln x)^2 \, dx$.

(e) $\int x^3 \cos(2x) \, dx$.

(f) $\int e^x \cos x \, dx$.

(g) $\int e^{2x} \sin(3x) \, dx$.

(h) $\int 2x e^{x^2} \cos(x^2) \, dx$.

Sometimes integration by parts may be applied in situations where the integrand does not look like a product at all. In such cases, the function $g'(x)$ is simply treated as the constant 1.

Example 5: $\int \ln x \, dx$.

The integrand can be written as $(\ln x) \cdot 1$, and thus we let $f(x) = \ln x$, $g'(x) = 1$. Then $f'(x) = \frac{1}{x}$, $g(x) = x$, and the formula for integration by parts gives:

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} x \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$$

Example 6: $\int \sin(\ln x) \, dx$.

The integrand can be written as $f(x)g'(x)$, where $f(x) = \sin(\ln x)$ and $g'(x) = 1$. Then $g(x) = x$, and be using the Chain Rule we find that $f'(x) = \cos(\ln x)\frac{1}{x}$. Now integration by parts gives the following:

$$\int \sin(\ln x) \, dx = x \sin(\ln x) - \int x \cos(\ln x) \frac{1}{x} \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx.$$

The integrand on the left hand side can be written as $f_1(x)g_1'(x)$, where $f_1(x) = \cos(\ln x)$ and $g_1'(x) = 1$. We have $f_1'(x) = -\sin(\ln x)\frac{1}{x}$ and $g(x) = x$. Thus

$$\int \cos(\ln x) \, dx = x \cos(\ln x) - \int x(-\sin(\ln x)) \frac{1}{x} \, dx = x \cos(\ln x) + \int \sin(\ln x) \, dx.$$

Substituting this result in the previous formula yields

$$\int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx.$$

Now we proceed as in Example 4 and add $\int \sin(\ln x) \, dx$ to both sides of the equation. This gives:

$$2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) + C,$$

and thus we finally get

$$\int \sin(\ln x) \, dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C.$$

Homework 3: Find the following integrals:

(a) $\int \arctan x \, dx$.
(b) $\int \arcsin x \, dx$.
(c) $\int \cos(\ln x) \, dx$.
(d) $\int (\ln x)^2 \, dx$.
(e) $\int e^{\sqrt{x}} \, dx$. *Hint:* Let $g'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$.

The formula for integration by parts also has an analogue for definite integrals:

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x) \, dx.$$