Math 442/542

Prelim 1 – Solutions

1. (a) The basic variables are x_1 and x_4. The non-basic variables are x_2 and x_3. The current basic feasible solution is $(x_1, x_2, x_3, x_4) = (2, 0, 0, 5)$ with objective function value 6.

   (b) In this tableau, the simplex method will choose x_2 as the entering variable and x_4 as the leaving variable. If we choose the correct entering variable (which is x_2) but the wrong leaving variable x_1, then the resulting basic solution becomes infeasible. This is because the minimum ratio rule will be violated. If we choose x_3 as the entering variable and x_1 as the leaving variable, then the resulting basic solution will be feasible (because the minimum ratio rule was applied) but it will have an objective function value which is less than that of the current one (because the coefficient of x_3 in row 0 is positive).

   (c) After one iteration of the simplex method, we get the following tableau:

	x_1	x_2	x_3	x_4	
z	0	0	4	1	11
x_1	1	0	$\frac{4}{5}$	$-\frac{1}{5}$	1
x_2	0	1	$\frac{1}{5}$	$\frac{1}{5}$	1

   The basic variables are x_1 and x_2. The non-basic variables are x_3 and x_4. The current basic feasible solution is $(x_1, x_2, x_3, x_4) = (1, 1, 0, 0)$ with objective function value 11. The current bfs is optimal since all the entries in row 0 are nonnegative.

2. Let x_{ij} be the number of units of size j product ($j = L, M, S$) produced at plant i ($i = 1, 2, 3$).

   The objective is to maximize

$$Z = 420 \sum_{i=1}^{3} x_{iL} + 360 \sum_{i=1}^{3} x_{iM} + 300 \sum_{i=1}^{3} x_{iS}$$

   subject to the following constraints:
   Capacity constraints:

$$\begin{aligned} x_{1L} + x_{1M} + x_{1S} &\le 750 \\ x_{2L} + x_{2M} + x_{2S} &\le 900 \\ x_{3L} + x_{3M} + x_{3S} &\le 450 \end{aligned}$$

   Storage space constraints:

$$\begin{aligned} 20x_{1L} + 15x_{1M} + 12x_{1S} &\le 13000 \\ 20x_{2L} + 15x_{2M} + 12x_{2S} &\le 12000 \\ 20x_{3L} + 15x_{3M} + 12x_{3S} &\le 5000 \end{aligned}$$

Same capacity percentage constraints:

$$900(x_{1L} + x_{1M} + x_{1S}) - 750(x_{2L} + x_{2M} + x_{2S}) = 0$$
$$450(x_{2L} + x_{2M} + x_{2S}) - 900(x_{3L} + x_{3M} + x_{3S}) = 0$$

Nonnegativity constraints:

$$\text{All} \quad x_{ij} \geq 0$$

3. (a) The associated basic feasible solution is $x = (1, 0, 2, 0, 0)$ with objective function value $z = 8$. Note that this solution is optimal since the entries in row 0 of the current tableau are all nonnegative.

(b) The optimal solution in (a) is not unique because (i) the coefficient of nonbasic variable x_2 in row 0 is 0; (ii) and all the entries in the column of x_2 are nonpositive. This means that we can increase x_2 infinitely without violating any constraints while keeping the objective function at the optimal value. This corresponds to having an optimal ray for the problem.
Let $x_2 = \lambda$. Then (considering that the other nonbasics stay at 0)

$$x_3 = 2 + \frac{1}{2}x_2 = 2 + \frac{1}{2}\lambda \quad \text{and} \quad x_1 = 1$$

and so, any solution of the form

$$x = (1, 0, 2, 0, 0) + \lambda(0, 1, \frac{1}{2}, 0, 0)$$

is optimal. Setting $\lambda = 1$ and $\lambda = 2$ yields the solutions

$$x = (1, 1, \frac{5}{2}, 0, 0) \quad \text{and} \quad x = (1, 2, 3, 0, 0).$$

Note that there are no other optimal *basic feasible* solutions since x_2 is the only non-basic variable with a zero coefficient in row 0 and it can be increased infinitely without forcing any of the basic variables to drop to 0 and leave the basis.