Math 442/542

Prelim 2 Solutions

1. (a) The basic variables are $\{x_4, x_2, x_1\}$ in that order. Thus,

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{pmatrix}, \; B^{-1} = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

Note that the columns of slack variables in the optimal tableau form B^{-1} based on the fundamental insight.

(b) The optimal solution to the dual problem can be read in row 0 of the optimal tableau: $(0, 2, 1, 0, 0, 2)$.

(c) The row-0 coefficient of x_3 in the current tableau is

$$c_B B^{-1} A_3 - c_3 = (0, 7, 4) \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} - c_3 = 3 - c_3 = 2.$$

So $c_3 = 3 - 2 = 1$.

(d) Suppose b_3 is changed to 7. Then the new right-hand side is

$$B^{-1} b = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix}$$

Thus the new values of basic variables: $(x_4, x_2, x_1) = (7, 3, -1)$. Since there is a negative entry in the righ-hand side (while row 0 is not changed), we use the dual simplex method to reoptimize. x_1 leaves the basis (the only basic variable with negative value) and x_5 enters (based on min-ratio test).

(e) Let x_7 be the new variable corresponding to the new product. Then its column in the constraint matrix is $A_7 = (2, 1, 4)$ and the profit/unit is its coefficient c_7 in the objective function. Since x_7 can be considered as a nonbasic variable in the final tableau, its row-0 coefficient in that tableau is:

$$c_B B^{-1} A_7 - c_7 = (0, 2, 1) \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - c_7 = 6 - c_7.$$

To make it worth producing, the current row-0 coefficient should be negative in which case x_7 will enter the basis. Thus, c_7 should be at least 6.

2. (a) The dual problem is

$$\begin{aligned} \max\ & 4y_1 - 10y_2 + 5y_3 \\ \text{s.t. } & 2y_1 + 2y_3 = 6 \\ & 2y_1 - 2y_2 + y_3 \le 4 \\ & -y_1 + 4y_2 - 3y_3 \ge -7 \\ & y_1 \ge 0, y_2 \le 0, \quad y_3 \text{ unrestricted} \end{aligned}$$

(b) If the solution is optimal, then the complementary slackness conditions must hold. We would like to find a complementary dual solution. We plug the values of (x_1, x_2, x_3) into the primal constraints and find that the second constraint is not tight. Thus, $y_2 = 0$. The complementary dual solution should have the same objective function value as the given primal solution which is $6 * (-2) + 4 * 3 - 7 * (-2) = 14$. So by solving the following system of equations:

$$4y_1 + 5y_3 = 14$$
$$2y_1 + 2y_3 = 6$$

we get $(y_1, y_2, y_3) = (1, 0, 2)$. We can easily check that this solution is dual feasible. Since both the primal and dual solutions are feasible with the same objective function value, our guess for the primal solution is optimal.

3. (a) First we need to make sure that the current tableau is optimal; so take $e \geq 0$ and $g \geq 0$. For the dual to have multiple optimal solutions, the primal optimal solution should be degenerate. To have that, take $g = 0$.

Summarizing, take $e \geq 0$, $g = 0$. f and h can be any numbers.

(b) For the dual to be unbounded we need two things: the primal should be infeasible, and the dual should be feasible. Taking $e \geq 0$ will provide that the dual is feasible: we can read a dual feasible solution in row 0 of the primal tableau. Let's rewrite row 1 of the tableau in equation form:

$$x_3 + fx_4 = g$$

Since $x_3, x_4 \geq 0$, taking $f \geq 0$ will provide that the left-hand side of the equation is a nonnegative number. Thus, taking $g < 0$ will provide that the constraint is not satisfied for any nonnegative x's, and so the primal is infeasible.

Summarizing, take $e \geq 0$, $f \geq 0$, $g < 0$; h can be anything.