Nonlinear dynamics of mental processes: Emotion-cognition interaction

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Abstract

Emotion (i.e., spontaneous motivation and subsequent implementation of a behavior) and cognition (i.e., problem solving by information processing) are essential to how we, as humans, respond to changes in our environment. Recent studies in cognitive science suggest that emotion and cognition are sub-served by different, although heavily integrated, neural systems. Understanding the time-varying relationship of emotion and cognition is a challenging goal with important implications for neuroscience. We formulate here the dynamical model of emotion-cognition interaction that based on the following principles: (1) the temporal evolution of cognitive and emotion modes are captured by the incoming stimuli and competition within and among themselves (ecological principle); (2) metastable states exist in the unified emotion-cognition phase space; and (3) the brain processes information with robust and reproducible transients through the sequence of metastable states. Such a model can take advantage of the often ignored temporal structure of the emotion-cognition interaction to provide a robust and generalizable method for understanding the relationship between brain activation and complex human behavior. The mathematical image of the robust and reproducible transient dynamics is a Stable Heteroclinic Sequence (SHS), and the Stable Heteroclinic Channels (SHCs). These have been hypothesized to be possible mechanisms that lead to the sequential transient behavior observed in networks. We investigate the modularity of SHCs, i.e., given a SHS and a SHC that is supported in one part of a network, we study conditions under which the SHC will continue to function in the presence of interfering activity with other parts of the network.

1 Introduction

The emotion-cognitive behavior has its basis in dynamical coordination of many brain centers, which often participate in both emotion and cognitive activity (Fales et al., 2008; M. D. Lewis, 2005; Pessoa, 2008). Due to this overlap, emotion and cognition are integrated in the sense of being partly separable (Gray et al., 2002). There are three main mechanisms of cognitive-emotional interaction: (i) involvement of multiple brain centers, like amygdala and prefrontal cortex, in both emotion and cognitive networks (Adolphs, 2008; Phelps, 2006); (ii) the high degree of connectivity between different brain areas (Bechara et al., 2000); and (iii) rhythmic brain activity (electric brain oscillations) at certain frequencies that supports coherent interactions between anatomically distinct regions of the brain during cognitive tasks, which require attention, working memory, or sensory processing (see, for example (Kelso, 1995; Buzsaki, 2006; Buzsaki & Draguhn, 2004)). An intriguing aspect of emotion-cognition interaction is the cognitive control via emotions. On the neurobiological level, cognitive control may be attributed to prefrontal activity inhibiting relevant sub-cortical emotion processing regions.

Emotion and cognition are sequential dynamic processes resulting from interactions of different brain subsystems (circuits) and their coordination / synchronization in time (Scherer, 1993). Sequential steps of emotion-cognition interactions are directly relevant to action control, in terms of memory, decision-making, reasoning, attention, and emotion regulation (Reis et al., 2007). A predominant phenomenon governing the dynamics of the brain is that the nervous system is responsible for its internal regulation, i.e., the generation and distribution of the energy and memory resources between the emotions, thoughts and actions. Thus, emotions and cognition are active processes that result in specific functional changing of the brain organization in time and the dynamical brains response to environmental information. These processes are determined by the functional (not necessary synaptic) connections between brain areas or neural circuits that participate in the execution of cognitive functions and generation of emotions. At different segments or steps of temporal emotional or cognitive process, the participating networks may vary, thus the temporal structure of the different emotions in the brain also is different. In terms of the collective network activity, emotion and cognition are not just spatial but spatio-temporal patterns, which are very sensitive to external or internal stimulus events. This is the way for brain to solve the fundamental conflict between the finite number of functionally-relevant centers in the representation and the continuous spectrum of different emotions and the huge variety of cognitive functions to be expressed. Such stimulus-dependent encoding is the origin of the enormous brain capacity. The understanding of the temporal structure of spatio-temporal patterns and experimentally verifiable models of the emotion-cognition sequential dynamics are the key steps to enlighten specific functional relationship within multiple neuro-anatomical structures, which are responsible for specific emotional and psychiatric disorders. The dynamical system theory is a natural domain for the analysis of such a complex network of neural clusters working coherently in time. Two key experimental observations guide us towards a dynamical model: (i) the existence of metastable cognitive states, and (ii) transitivity of reproducible cognitive processes (Rabinovich, Huerta, Varona, & Afraimovich, 2008).

Despite the strong coupling between the cognition and emotions, their dependence on time can be quite different. The main difference is the following: the emotion may be quasi-static or recurrent in time, whereas a cognitive activity, by the nature of its task, must be transient in time until the termination of the executed cognitive function. It can then return back to a static or rhythmic regime. Nonlinear dynamics constitutes the only feasible medium that can accommodate such a behavior. The idea that emotion-cognitive activity can be understood using nonlinear dynamics has been intensively discussed at length for the last 15 years (M. Lewis et al., 2008; Friston, 1997, 2000; Friston et al., 2003; Port & Gelder, 1995).

The main problem faced when using dynamical systems theory to describe transient neural activity is the fundamental contradiction between reproducibility and flexibility of transient behavior (Vogels et al., 2005; Abott, 2008). A nonlinear dynamical behavior confined in a stable sequence of metastable states is the only feasible solution to this dilemma.

Metastability is a general nonlinear dynamics concept, which describes states of delicate equilibrium. A system is in a metastable state when it is in the vicinity of such an equilibrium, i.e., a state where the system spends an extended (but finite) period of time. Under the action of perturbations or interaction system is susceptible to fall into another state. Metastability in the brain is a phenomenon, which is being studied in neuroscience to elucidate how the human mind process information and recognizes patterns. There are semi-transient signals in the brain, which persist for a while and are different than the usual equilibrium state (Abeles et al., 1995; Werner, 2007). Thus, metastability is a principle that describes the brains ability to make sense out of seemingly random environmental cues (Oullier & Kelso, 2006). In the past 25 years, interest in metastability and the underlying framework of nonlinear dynamics has been fueled by advancements in the methods by which computers model brain activity. The metastability is supported by the flexibility of coupling among diverse brain centers or neuron groups (Friston, 1997, 2000; Ito et al., 2007; Sasaki et al., 2007): that is, in the form of a continuum of dynamically shifting, discrete configurations of brain networks (for a review, see (Fingelkurts & Fingelkurts, 2006)). The temporal order of the metastable states are determined by the functional connectivity of the underlying networks and their causality structure (Chen et al., 2009).The brain metastable states, in their turn, must appear in the EEG in the form of its piecewise stationary organization which can be studied by means of the change-point analysis (Kaplan & Shishkin, 2000).

We develop here a theoretical description of the transient emotion-cognitive dynamics based on the interaction of functionally-dependent emotion-cognitive modes. The basis of this model has been recently discussed in (Rabinovich, Huerta, Varona, & Afraimovich, 2008) in the context of cognitive modes competition without taking into account an emotional influence. The core of this paradigm is a sequential Winnerless Competition (WLC) of different cortical sub-networks for the brain resources along the execution process. Such competition is reminiscent of the competition of different species for the environmental resources in ecology. WLC principle was formulated first for the sensory systems (Rabinovich et al., 2001; Levi et al., 2004) and was then applied for the description of some cognitive function like decision-making (Rabinovich, Huerta, Varona, & Afraimovich, 2008). Recent experimental evidence pinpoints competition among metastable states in the rat gustatory cortex (Jones et al., 2007) also in olfactory system (Rabinovich, Huerta, & Laurent, 2008). A competitive and concurrent activity of multiple brain areas (Fox et al., 2005, 2007) that collaborate in a large-scale cortical process is fundamentally important for thinking, in particular, for sentence comprehension (Just & Varma, 2007). Physiological mechanism of such competition is the inhibition (see for a review (Aron, 2007)).

The paper is organized as follows: The winnerless competition principle and a unified model of such joint activity among brain modes is introduced in the following section, which also presents a simulation that accounts for possible qualitative and quantitative aspects of their interaction. In Section 3, the mathematical framework of this interference is presented and rigorous conditions on the robustness of metastable behavior are derived.

2 Ecological Principle in the Brain

Neural activity demands certain physical (e.g., glucose, oxygen, etc.) and informational (e.g., sensory stimulation) resources. A predominant factor underlying the brain dynamics is the competition among brain centers and modes for these finite resources. The fundamental mechanism in carrying out this competition is inhibition, which is widespread in the cortex. Under mild constraints, the multi-agent competition induces a *transient* and *predictable* dynamics. The SHC is a solid embodiment of these vital features.

2.1 Generalized Lotka-Volterra Equations

A widely-accepted model of competition among N parties is the Generalized Lotka-Volterra (GLV) system

$$\frac{dx_i}{dt} = x_i \left(\sigma_i(I) - \sum_{j=1}^{N} \rho_{ij}(I) x_j + \eta(t) \right), \quad i = 1, \ldots, N, \tag{1}$$

where $x_i \geq 0$ denotes the ith competitor, I summarizes all observable environmental factors that influence competition, $\sigma_i \geq 0$ is the resources available for the competitor i to prosper, ρ_{ij} is the competition matrix with non-negative entries, and η is a noise process, capturing all unpredictable effects from the environment.

As introduced in (Afraimovich, Zhigulin, & Rabinovich, 2004), the following conditions ensure a SHC of N saddles:

$$\frac{\sigma_{i-1}}{\sigma_i} < \rho_{i-1,i} < \frac{\sigma_{i-1}}{\sigma_i} + 1, \qquad 1 < i \leq N; \tag{2}$$

$$\frac{\sigma_{i+1}}{\sigma_i} - 1 < \rho_{i+1,i} < \frac{\sigma_{i+1}}{\sigma_i}, \qquad 1 \leq i < N; \tag{3}$$

$$\rho_{ji} > \rho_{i-1,i} + \frac{\sigma_j - \sigma_{i-1}}{\sigma_i}, \quad 1 < i \leq N, \ j < i-1, \ j > i+1. \tag{4}$$

The self-inhibition of each competitor is quantified by $\rho_{ii} = 1$, and the noise magnitude $|\eta|$ is assumed to be sufficiently small. Without loss of generality, these conditions place the N saddles on the N axes of the phase space, the ith one with σ_i distance from the origin, and order them along the SHC with respect to the index i.

2.2 Model of Emotion-Cognition Interaction

We represent the centers/modes participating in the cognitive activity by $A_i \geq 0$, $i = 1, \ldots, N$. Following the ecological principle discussed above, for a given cognitive load I, we adopt the GLV model (1) for the cognitive modes

$$\frac{dA_i}{dt} = A_i \left(\alpha_i(I, \mathbf{B}) - \sum_{j=1}^{N} \rho_{ij}(\mathbf{B}) A_j + \eta_A \right), \quad i = 1, \ldots, N, \tag{5}$$

where $\mathbf{B} = [B_1 \ \cdots \ B_M]$ denotes the emotional activity, and all other parameters are defined as in (1). We also assume that, in the absence of an emotional distractor (which may be a certain spatio-temporal pattern observed in $\mathbf{B}$), the cognitive activity follows a sequence of metastable equilibria, i.e., the cognitive trajectory $A_i(t)$, $i = 1, \ldots, N$, is confined in an SHC. In particular, α_i and ρ_{ij} satisfy the conditions (2)-(4) and $|\eta_A|$ is sufficiently small.

Generated by specific brain centers/modes subject to competition, the emotional activity is also considered to be varying according to GLV equations:

$$\frac{dB_i}{dt} = B_i \left(\zeta_i(S, \mathbf{A}) - \sum_{j=1}^{M} \xi_{ij}(\mathbf{A}) B_j + \eta_B \right), \quad i = 1, \ldots, M. \tag{6}$$

Here S denotes the emotional stimuli (or stressor), and all other parameters are defined as above. In contrast to the cognitive process, we do not restrict the emotion model to a heteroclinic dynamics; it can also demonstrate periodic or chaotic oscillations.

Equations (5) and (6) describe the joint emotio-cognitive dynamics. In the sequel, we limit ourselves to the special case where the term $\alpha_i(I, \mathbf{B})$ in (5) can be decomposed as $\sigma_i(I) + \epsilon \cdot \gamma_i(\mathbf{B})$ and the competition matrix ρ_{ij} is constant. With these assumptions, the joint dynamics fall under the winnerless competition model introduced in (9).

2.3 Simulation

We simulated $N = 5$ cognitive and $M = 5$ emotional modes that evolve in periodic (closed) SHCs using the GLV equations (5) and (6). The simulation focused on the effect of emotional process on the cognitive process, thus we further simplified the emotion model by selecting $\zeta_i(S, \mathbf{A}) = \zeta_i(S) = S$, and ξ_{ij} as independent of $\mathbf{A}$. The cognitive resource component $\sigma_i(I)$ is selected as the unity.

The constant competition matrices ρ_{ij} and ξ_{ij} are set based on the conditions (2)-(4), with σ_i substituted by the unity in all inequalities. Specifically, we evaluated $\rho_{i-1,i}$ and $\rho_{i+1,i}$ at the mid-points of the the intervals imposed in (2) and (3), and assigned $\rho_{ji} = \rho_{i-1,i}+0.5$ to satisfy (4). This establishes the periodic stable heteroclinic sequence $e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_5 \rightarrow e_1$, e_i denoting the ith unit vector, in each of the emotion and the cognitive working (phase) spaces, when the resource terms α_i and ζ_i equal 1 for all i.

Our setup assigns to certain emotional modes, namely B_1 and B_2, a *distracting* role on the sequential behavior on the A network, whereas B_4 is a *motivating* emotional mode. We implement this through the coupling term $\gamma_i(\mathbf{B}) = [0.5\ 0.5\ 1.0\ 2.0\ 1.0]^T \cdot \mathbf{B}$ for $i = 1, \ldots, 5$. With this interaction scheme, the distracting emotional modes slow-down the cognitive network's switching pattern in time and reduce the cognitive modes magnitude. In the extreme distraction, the order of cognitive modes is disrupted.

We set the emotion process to evolve on a slower time scale, thus the noise terms η_A and η_B were selected as white noise processes with variances 10^{-8} and 0.005, respectively.

3 Mathematical Framework

3.1 The general setup

Consider a system of differential equations:

$$\frac{dx}{dt} = f(x), \tag{7}$$

where $x \in \mathbb{R}^d$. This system is said to possess a *stable heteroclinic sequence* (SHS) if it has a finite sequence $\{Q_1, Q_2, ..., Q_n\}$ of hyperbolic equilibrium points such that the unstable

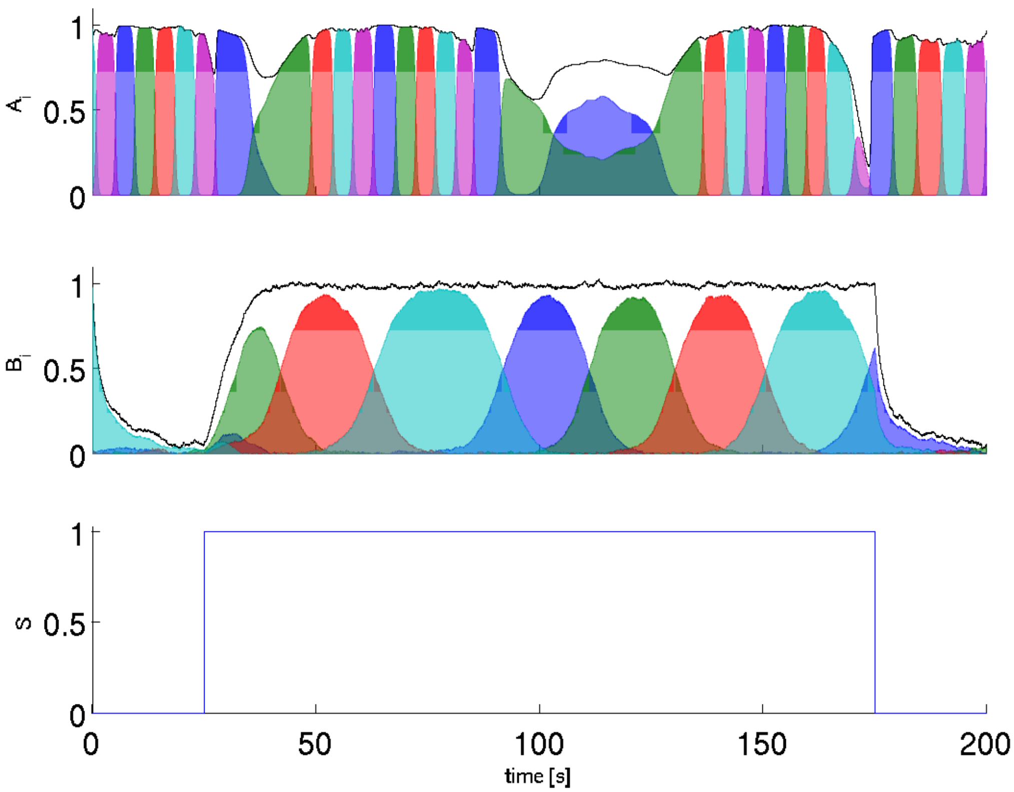

Figure 1: Simulation of the emotion-cognitive model with the parameters listed in the text. According to the selected interaction scheme $\gamma(\mathbf{B})$, the emotion modes B_1 and B_2 (indicated by the blue and green curves in the second row) are capable of derailing the A network from the cognitive SHC. When the coupling ϵ is weak (as in the early and the late phases of the stressor cycle, the emotion modes B_1 and B_2 still inhibit (slow down) the cognitive process), but the cognitive network does not loose its prescribed track.

manifold for x_i intersects the stable manifold of x_{i+1}, for each $1 \leq i < n$, in a particular way that we will make precise below. The union of the equilibrium points along with the intersections of their stable and unstable manifold is the heteroclinic sequence. Under certain conditions (Afraimovich et al., 2004) there is an open set O in a neighborhood of a SHS such that the orbit starting at any initial condition inside O will continue inside O until it emerges near x_n. This open set O is called a *stable heteroclinic channel* (SHC) and it is shown to be robust under small perturbations of the vector field (as defined by the differential equation).

In addition to the SHS in the components of the system represented by the x variables, we suppose that the network has many other components, represented by the vector variable y, and that these components are weakly coupled to the portion of the network that in isolation would support a SHS. That is, the entire network satisfies equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= f(x) + \epsilon g(x, y) \\ \frac{dy}{dt} &= h(x, y), \end{aligned} \tag{8}$$

where for this f, equation (7) possesses a SHS. In this model we will employ Normal Hyperbolicity, which imposes restrictions on h, to prove our results.

A special case of coupling is the competition models:

$$\begin{aligned} \frac{dx}{dt} &= (f(x) + \epsilon g(x, y))x \\ \frac{dy}{dt} &= h(x, y) \end{aligned} \tag{9}$$

We will denote by Φ^t the flow generated by (8) or (9) depending on the context. We will use the notation ϕ^t to denote the flow generated by the unperturbed, subsystem (7).

In more realistic models, we might suppose f is a function of both x and y and that for y in certain regions of phase space the system supports a SHS in x, but for y in other regions, there is no SHS in the x variable. This model would allow for the possibility of "switching" the function associated with the SHC on or off, but is beyond the scope of the current study.

3.2 Stable Heteroclinic Sequences and Channels

We will assume that (7) possesses a SHS. In this section we will make this and several preliminary ideas precise.

At the i-th saddle point Q_i we suppose that the eigenvalues of the linearization of (7) can be ordered:

$$\lambda_1^{(i)} > 0 > \operatorname{Re} \lambda_2^{(i)} \geq \operatorname{Re} \lambda_3^{(i)} \geq \ldots \geq \operatorname{Re} \lambda_d^{(i)}.$$

Thus each Q_i possesses a $d-1$ dimensional stable manifold W_i^s and a 1 dimensional unstable manifold that consists of two separatrices, Γ_i^+ and Γ_i^-.

Definition 3.1 *We will say that* $\{Q_i\}_{i=1}^N$ *belongs to a heteroclinic sequence, if* $\Gamma_i^+ \subset W_{i+1}^s$ *for each* $1 \leq i < N$.

Definition 3.2 *The number*

$$\nu_i = \frac{-Re\,\lambda_2^{(i)}}{\lambda_1^{(i)}}$$

is called the saddle value *for* Q_i. *If* $\nu_i > 1$, *then the saddle* Q_i *is called* dissipative. *If all the saddles in a heteroclinic sequence are dissipative, then we call*

$$\Gamma = \bigcup_{i=1}^{N} Q_i \bigcup_{i=1}^{N-1} \Gamma_i^+$$

a Stable Heteroclinic Sequence.

3.3 A SHC embedded in a larger system

For both models (8) and (9), we will assume that $h(x, y)$ is dissipative in y in the sense that for each x, the system:

$$\frac{dy}{dt} = h(x, y), \qquad x \text{ fixed} \tag{10}$$

contracts volume in phase space and has a *ball of dissipation*, i.e. a bounded ball $B \subset \mathbb{R}^d$ such that any orbit in $\mathbb{R}^d$ eventually enters B and no orbits in B escape from B in forward time. Under these conditions (10) has a maximal compact attractor Λ_x for each x. In particular, Λ_{Q_i} will denote the maximal attractor for $x = Q_i$. This implies that for $\epsilon = 0$ and each Q_i the system (8) has a compact saddle set given by $\mathit{\Lambda}_i = Q_i \times \Lambda_{Q_i}$. However, we will not assume that $\mathit{\Lambda}_i$ is structurally stable.

4 General Coupled Systems

4.1 Normal Hyperbolicity

Let V be a smooth (C^∞) compact sub-manifold of a smooth manifold M. Suppose $f : M \to M$ is a C^1 diffeomorphism and $f(V) = V$. Suppose $T_V M$, the tangent bundle of M over V, have a Df-invariant splitting

$$T_V M = E^u \oplus TV \oplus E^s.$$

At each $p \in V$ set

$$V_p f = Df|_{T_p V}, \qquad E_p^u f = Df|_{E_p^u} \qquad E_p^s f = Df|_{E_p^s}.$$

Definition 4.1 *(Hirsh et al., 1977) The diffeomorphism* f *is* (immediately relatively) r-normally hyperbolic at V *if* f *is* C^r *and there is a Riemann structure on* TM *such that for all* $p \in V$, *and* $0 \leq k \leq r$:

(a) $m(E^u_p f) > \|V_p f\|^k$, *and*

(b) $\|E^s_p f\| < m(V_p f)^k$,

where $m(A)$ *is the* minimum norm *of a linear transformation* A*, i.e.*

$$m(A) = \inf\{|Ax| : |x| = 1\|.$$

We will use simply the term r-normally hyperbolic to denote immediate relative normal hyperbolicity.

If $Q \in V$ is a fixed point and $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q, \gamma_1, \ldots, \gamma_r$ are eigenvalues of $Df|_Q$ such that the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_p$ belong to E^s, the eigenvectors corresponding to $\mu_1, \ldots, \mu_q$ belong to TV, and those corresponding to $\gamma_1, \ldots, \gamma_r$ belong to E^u, then the conditions of r-normal hyperbolicity can be rewritten as

$$\begin{aligned} \min|\gamma_i| &> \max|\mu_j|^k, \\ \max|\lambda_i| &< \min|\mu_j|^k, \end{aligned} \tag{11}$$

for $0 \leq k \leq r$.

If Q is an equilibrium of a flow, $Q \in V$, then the conditions for 1-normal hyperbolicity can be written in the following form

$$\begin{aligned} \min \mathrm{Re}\gamma_i &> \max \mathrm{Re}\mu_j, \\ \max \mathrm{Re}\lambda_i &< \min \mathrm{Re}\mu_j, \end{aligned} \tag{12}$$

where $\{\gamma_i\}_{i=1}^p$, $\{\mu_i\}_{i=1}^q$, and $\{\lambda_i\}_{i=1}^r$ are the eigenvalues of the linearized system at the point Q corresponding to E^u, TV, and E^s, respectively.

The following is the Fundamental Theorem of Normally Hyperbolic Invariant Manifolds.

Theorem 4.2 *(Hirsh et al., 1977) If f is r-normally hyperbolic at V, then through V pass stable and unstable invariant manifolds $W^s(V)$ and $W^u(V)$ respectively which are tangent at V to $TV \oplus E^s$ and $E^u \oplus TV$. They are of class C^r. The stable manifold is invariantly fibered by C^r sub-manifolds tangent at V to the subspaces E^s (these are W^s_p.) Similarly for the unstable manifold and E^u. These structures are unique and persistent under small perturbations of f. Similar results hold for flows.*

4.2 Normal Hyperbolicity in the Model

We wish to apply normal hyperbolicity to each ($\epsilon = 0$) invariant manifold $\Lambda_i = Q_i \times Y$. These manifolds are smooth and the tangent space has an invariant splitting there given by

$$(E^u_{Q_i} \times Y) \oplus T\Lambda_i \oplus (E^s_{Q_i} \times Y)$$

which the flow corresponding to $\epsilon = 0$ is expanding along $E^u_{Q_i} \times Y$ and contracting along $E^s_{Q_i} \times Y$. However, there are two missing ingredients to normal hyperbolicity: (1) $\Lambda_i = Q_i \times Y$ is not compact and (2) the action of Φ^t along $T\Lambda_i$ needs to be specified to satisfy the inequalities required in (4.1).

Obstacle (1) is easily overcome by the standard device of a *cutoff function*. Suppose the union of some balls of dissipation for each x is contained in a y ball of radius R. Denote this ball by B_R. Then let $\chi(y)$ be a smooth function with is 1 on B_R, 0 outside the ball of radius $R+1$ and $0 < \chi(y) < 1$ on the set $B_{R+1} \setminus B_R$. Now if we multiply $h(x, y)$ by χ and restrict the system to the closed $R+1$ ball in y, then we have that each $Q_i \times \overline{B_{R+1}}$ is invariant and compact. The dynamics of this system inside the absorbing set is exactly the same as the previous system.

The second obstacle (2) requires additional assumptions on the flow in the y direction. For our purposes 1-normal hyperbolicity, i.e. $r = 1$ will be sufficient. If Φ^t is a flow, normal hyperbolicity means 1-normal hyperbolicity for some t for the map Φ^t. That is, setting $\Lambda^i = Q_i \times B_{R+1}$, the flow must satisfy:

(a) $m(E^u_p \Phi^t) > \|\Lambda^i_p \Phi^t\|$, and

(b) $\|E^s_p \Phi^t\| < m(\Lambda^i_p \Phi^t)$,

at each $p \in \Lambda^i$ for some $t > 0$.

One would wish to define normal hyperbolicity in terms of the vector field (differential equations) defining the flow, however, as pointed out in (Hale, 1969) and elsewhere, such conditions are subtle.

Under the condition that each $Q_i \times B_{R+1}$ is normally hyperbolic in the sense of (Hirsh et al., 1977), then these invariant manifolds persist as normally hyperbolic invariant manifolds for $\epsilon > 0$. It also follows from Theorem 4.2 that the stable and unstable manifolds $W^s_i \times B_{R+1}$ and $\Gamma^+_i \times B_{R+1}$ of $Q_i \times B_{R+1}$ depend smoothly on ϵ for $\epsilon \geq 0$.

By using the inequalities (11), one may impose conditions under which the system (8) (or (9)) will be normally hyperbolic in a neighborhood of an equilibrium. Assume that (8) for $\epsilon = 0$ has an equilibrium $Q_i \times P$ at the coordinates $x = x^*$, $y = y^*$, such that $\lambda_1, \ldots, \lambda_p, \gamma_1, \ldots, \gamma_r \in \text{spec } Df|_{Q_i}$; $\text{Re}\lambda_j < 0$, $j = 1, \ldots, p$; $\text{Re}\gamma_j > 0$, $j = 1, \ldots, r$, and $\mu_1, \ldots, \mu_q \in \text{spec } \frac{\partial h(x^*, y^*)}{\partial y}$. If the conditions (12) are satisfied, then the system (8) is normally hyperbolic at the point $Q_i \times P$ and within a small neighborhood of it.

Now, within each $Q_i \times B_{R+1}$ and for $\epsilon = 0$ there is an attractor Λ_{Q_i}. For $\epsilon > 0$, while Λ_{Q_i} may not persist, it is nonetheless succeeded by a new maximal attractor inside the new invariant manifold. These maximal attractors have some semi-continuity property with respect to ϵ, but these features are not needed in this article.

We denote by $W^{cs}_i(\epsilon)$ the central stable normally hyperbolic manifold, $W^{cs}_i(0) = W^s_i \times B_{R+1}$, and by $\Gamma^{cu}_i(\epsilon)$ the piece of central unstable manifold, $\Gamma^{cu}_i(0) = \Gamma^+_i \times B_{R+1}$. Let $\Gamma^+_i(\epsilon)$

($W_i^+(\epsilon)$) be the projection of $\Gamma_i^{cu}(\epsilon)$ ($W_i^{cu}(\epsilon)$) on the x-space.

Since locally $\Gamma_i^+(\epsilon)$ and $W_i^s(\epsilon)$ depend smoothly on ϵ, a compact portion of $\Gamma_i^+(\epsilon)$ will remain close to $W_{i+1}^s(\epsilon)$. Thus, the x-projections of the orbits leaving a neighborhood of $Q_i \times B_{R+1}$ will approach Q_{i+1}. Locally near Q_{i+1} the x-projections of the orbit will follow Γ_{i+1} away from Q_{i+1} and, hence, the x-projections of the orbit will continue to follow close to the original SHS.

Theorem 4.3 *Under the assumption of partial hyperbolicity for the system (8), if system (7) has a SHS then there exists an open set in the Banach space of vector fields (8) such that for $\epsilon > 0$ sufficiently small the full system has a SHC. In this context this means: Given any neighborhood of the SHS, there exists an open set O such that any orbit starting from an initial condition in $O \times B_R$ will remain in $O \times B_R$ until it leaves this set in a neighborhood of $Q_n \times B_R$.*

Note that orbits described in the theorem when projected onto the x directions will come close to Q_1, Q_2, ..., Q_n in order, thus executing the sequential pattern of activation associated with the original SHS in the isolated system.

To derive sufficient conditions (in terms of the vector functions g and h) that cover all (or as many as possible) specific cases, one needs to apply a special technique. One has to study the behavior of integrals of perturbations taken along heteroclinic orbits of the unperturbed system, i.e., to derive and study Melnikov-type functions. Due to the tediousness of and technical difficulties with this approach, we find it more suitable here to focus on a special class of perturbations, namely the ones in the form

$$g(x, y) = g_0(x) + \delta g_1(x, y), \quad \delta \ll 1, \tag{13}$$

where h is an arbitrary smooth vector function.

Proof: For the class of perturbations (13), it was shown in (Rabinovich, Huerta, Varona, & Afraimovich, 2008) that, if the system (7) has a SHS containing the saddles $Q_1, \ldots, Q_n$, then the system

$$\dot{x} = f(x) + \epsilon g_0(x) \tag{14}$$

has a SHC joining neighborhoods of these saddles for a given small $\epsilon > 0$ and an open set $\{g_0\} := Z$ of vectors in the Banach space of the vector fields with the C^1 norm. It follows from the construction in (Rabinovich, Huerta, Varona, & Afraimovich, 2008) that one can choose an open set U of initial points in a neighborhood of Q_1 in such a way that every trajectory of (14) starting at $x_0 \in U$ contains a segment Γ joining x_0 with a point, say, x_n in a neighborhood of Q_n such that

(i) Γ belongs to a SHC,

(ii) $\mathrm{dist}(\Gamma, Q_i) > \rho_0 > 0$, where ρ_0 is a number independent of $i = 1, \dots, n$, and independent of the choice of $g_0 \in Z$. It follows that the temporal length $\ell(\Gamma)$ of $\Gamma < R_0$, where R_0 is a number independent of the choice of initial point $x_0 \in U$, and $g_0 \in Z$. Then the direct corollary of the theorem on the smooth dependence of an ODE solution on parameters is the following fact: given $g_1(x, y)$ and $h(x, y)$, there is $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$, the x-projection of a solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of the system (8) with $g = g_0(x) + \delta g_1(x, y)$ and $x_0 \in U$, $|y_0| \le R$ of temporal length equals $\ell(\Gamma)$ will be C^1-close to Γ, uniformly with respect to $g_0 \in Z$, $x_0 \in U$, $|y_0| < R$, and the piece of the x-projection goes to Γ (in C^1-norm) as $\delta \to 0$. Thus, one can choose δ to be so small that this piece will stay in a SHC until it reaches a neighborhood of Q_n. □

Remark 1 As one can see, for the class Z of perturbations in (14), the assumption of partial hyperbolicity is not needed. Of course, the set Z of perturbations indicated therein is small and generally "non-practical."

Remark 2 *About the proof in general case.* Because of the partial hyperbolicity assumptions, the piece $\Gamma_i^+(\epsilon)$ is close to Γ_i^+ and, thus, will approach a neighborhood, say, V_{i+1} of Q_{i+1}. The local piece of $W_{i+1}^s(\epsilon)$ divides V_{i+1} into two parts: the containing $\Gamma_{i+1}^+(\epsilon)$ will be denoted by V_{i+1}^+. There are two possibilities: (1) $\Gamma_i^+(\epsilon)$ enters V_{i+1}^+, or (2) it enters $V_{i+1} \setminus V_{i+1}^+$. Each of these corresponds a different class of vector functions $g(x, y)$, say, $S_{(1)}$ and $S_{(2)}$. The main problem is to describe the conditions for the occurrence of the class $S_{(1)}$ in a suitable form for $i = 1, \dots, n$, since, if the situation $S_{(2)}$ holds, then the x-coordinate of an orbit with an initial point close to $\Gamma_i^+(\epsilon)$ can follow Γ_{i+1}^- preventing the realization of SHC.

The important point is: in spite of the form of conditions for the occurrence of $S_{(1)}$, these conditions are "open," i.e., they single out an open set in the Banach space of vector fields.

In the case of $S_{(1)}$, the x-coordinate of an orbit will follow $\Gamma_{i+1}^+(\epsilon)$. This can be shown in the same way as followed in (Rabinovich, Huerta, Varona, & Afraimovich, 2008). A piece of $\Gamma_i^+(\epsilon)$ has an end point, say, $x_{i+1} \in V_{i+1}^+$ with $\text{dist}\left(x_{i+1}, W_{i+1}^s(\epsilon)\right) < K_{i+1} \cdot \epsilon$, K_{i+1} being a a constant independent of ϵ. Given an initial point (x_0, y_0) with $x_0 \in V_{i+1}^+$, $\text{dist}\,(x_0, x_{i+1}) < \epsilon$, the corresponding solution $(x(t), y(t))$ of (8) at the instant of exit from V_{i+1}^+ satisfies the inequality

$$\text{dist}\left(x(t), \Gamma_{i+1}^+(\epsilon)\right) < (K_{i+1} \cdot \epsilon)^{\mu_{i+1}} \tag{15}$$

where $\mu_{i+1} = \nu_{i+1} - \alpha(\epsilon)$, $\alpha(\epsilon)$ is small and ν_{i+1} is the saddle value of Q_{i+1}, so $\mu_{i+1} > 1$. The validity of (15) can be shown by a local study of solutions of (8) in a neighborhood of a saddle invariant subset, similar to the considerations (Shilnikov et al., 1998, 2001). To proceed in the study from $i + 1$ to $i + 2$, one must be sure that

$$(K_{i+1} \cdot \epsilon)^{\mu_{i+1}} < \epsilon. \tag{16}$$

The condition (16) also singles out an open set in the Banach space of vector fields (8).

For perturbations in the form (14), one can express such conditions (for $S_{(1)}$) by studying the system first for $\delta = 0$ by using the methods of (Rabinovich, Huerta, Varona, & Afraimovich, 2008), and then for a small $\delta > 0$ by applying the theorems of smooth dependence on parameters. Nevertheless, the study becomes too technical and we skip this line in this article.

Remark 3 In many applications each x_i represents the activation level of some neuron or cluster of neurons or mode oscillation, etc. and the phase space is a $\mathbb{R}^n_+ = x : x_i \geq 0$. In this situation it is natural that each saddle Q_i is on the boundary of phase space and Γ^- points outside. Thus orbits entering a neighborhood of Q_i cannot follow Γ_i^- away from Q_i, and so must follow Γ^+, i.e., the case $S_{(1)}$ occurs automatically.

4.3 Partial hyperbolicity conditions in a neighborhood of equilibrium

First of all, let us emphasize that the dissipativeness of the system (6), for any fixed $\mathbf{A}$ and in the absence of noise follows directly from the assumption $\xi_{ij}(\mathrm{A}) > 0$.

In the absence of noise, the system (11) for $\epsilon = 0$ has an equibrium $Q_i : \mathbf{A}^{(i)} = (0 \ \cdots \ 0 \ \sigma_i \ 0 \ \cdots \ 0)$ with characteristic exponents $-\sigma_i$ and $\sigma_j - \rho_{ji}\sigma_i$, $j = 1, \dots, N$, $j \neq i$. Furthermore, for $\mathbf{A} = \mathbf{A}^{(i)}$, the system (12), in the absence of noise, has equilibria $P_{i,k}$: $B^{i,k} = \big(0 \ \cdots \ 0 \ \zeta_k(S, \mathbf{A}^{(i)}) \ 0 \ \cdots \ 0\big)$, with characteristic exponents $-\zeta_{k,i} := -\zeta_k(S, \mathbf{A}^{(i)})$ and $\zeta_{k,i} - \xi_{k,m}\zeta_{m,i}$, $m = 1, \dots, M$, $m \neq k$.

We assume that Q_i belongs to an SHS, so it has the one-dimensional unstable manifold, i.e., only one among the numbers $\{\sigma_i - \rho_{ij}\sigma_i\}$ is positive. Let $\sigma_{j_0} - \rho_{j_0 i}\sigma_i > 0$. The conditions (12) of partial hyperbolicity in a neighborhood of the equilibria $Q_i \times P_{i,k}$ can be rewritten as follows:

$$\sigma_{j_0} - \rho_{j_0 i}\sigma_i = \max_j \sigma_j - \rho_{ji}\sigma_i > \max_{k,m} \zeta_{k,i} - \xi_{k,m}\left(\mathbf{A}^{(i)}\right)\zeta_{m,i} \tag{17}$$

$$\max_{j, j \neq j_0, i} \sigma_j - \rho_{ji}\sigma_i < \min_{k,m} \zeta_{k,i} - \xi_{k,m}\left(\mathbf{A}^{(i)}\right)\zeta_{m,i}. \tag{18}$$

Of course, the conditions are only sufficient, but if they are not satisfied, then it might happen that the SHC would not persist for $\epsilon \neq 0$. Numerical study shows that, for a wide set of initial points, the system spent plenty of time in neighborhoods of equilibria, so it is natural to assume that the conditions (17) and (18) would imply partial hyperbolicity for a large variety of solutions.

5 Conclusions and Perspectives

We have presented a coupled emotion-cognitive model based on the ecological principles that govern brain centers. The model and rigorous mathematical results about the persistence of SHC are general enough and open new alleys in the quantitative theory of emotion-cognition interaction. We wish to formulate here a few of them:

1. It is experimentally justified that emotions are represented as the superposition of slow and pulsating components. According to our model, these parts may have distinct roles on different aspects of the cognitive activity. As our preliminary results show, the vibrating part of emotion acts as a multiplicative noise on the cognition vector field. This is consequential on the exit time from metastable states, i.e., the maximal time the system spends in the vicinity of saddles (see (Kifer, 1981; Stone & Holmes, 1990) for a thorough treatment of this aspect). In fact, a local stability analysis in the vicinity of a saddle fixed point allows one to estimate the this time in the light of the relation $t_e = 1/l \cdot \ln 1/|g|$ where t_e is the mean exit time, $|g|$ is the level of emotion pulsation, and l is an eigenvalue corresponding to the unstable separatrix of the saddle. Thus, the vibrating part of emotion, unless it is too large in magnitude, facilitates the execution of the cognitive problem as it accelerates the "stream of thought" along the SHC. Too strong emotion pulsation would move the cognitive system out of SHC. It is an important problem to estimate this critical value of emotional vibrations.

2. The slowly changing in time part of emotions (for example, chronic depression) is able to change the stability structure of the heteroclinic channel because it changes the effective increment σ. This part can generate saddles with multi-dimensional unstable manifolds. Although this scenario may seem inconsequential for the system performance (since the unstable direction with the maximum eigenvalue is likely to be followed on exit from the saddle), in reality, each exit direction would be assigned a certain probability for being followed by the system. We hope that it would be possible to formulate the relationship between this probability distribution and the eigenvalues.

3. In this article, we did not emphasize the fact that the **B**-system behaves also in a sequential manner and did not study the relation between the switching instants between metastable states in any of the systems. The synchronization of multiple SHCs in coupled neural networks is an important problem that will have both theoretical and practical consequences.

It seems that the partial hyperbolicity approach works adequately in the considered situations. Conditions of partial hyperbolicity, for instance (17) and (18), enlightens the boundaries of the instability in the emotion system to follow a prescribed cognitive function.

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